Interpolatory Surface Subdivision based on Geometry-Driven Parameterizations

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Outline

Subdivision surfaces

Basic idea:

- An initial 2-manifold network of vertices, edges and facets - called *control mesh* or *control polyhedron* - is refined by computing new vertices and joining them up to form a new mesh with the same connectivity.
- By repeating recursively this operation (if the refinement construction is suitable), the mesh converges towards a *limit surface*.

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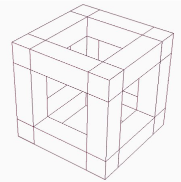

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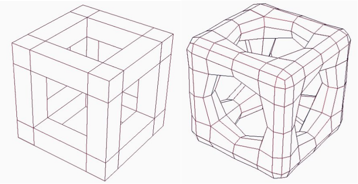

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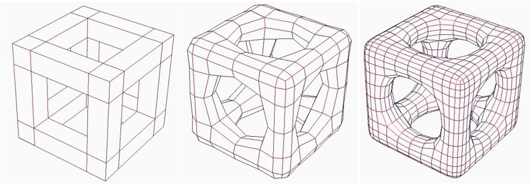

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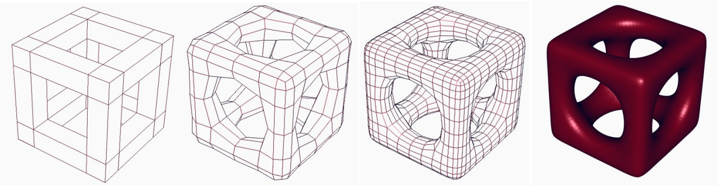

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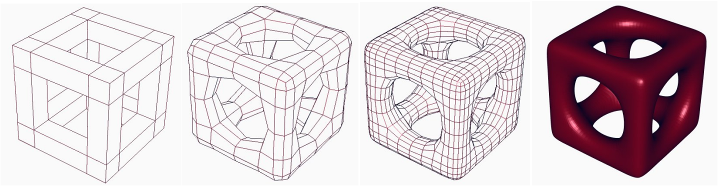

Main advantage over splines: subdivision naturally handles meshes of *arbitrary topology* (i.e. made of any number of quadrilateral or triangular facets meeting at a vertex).

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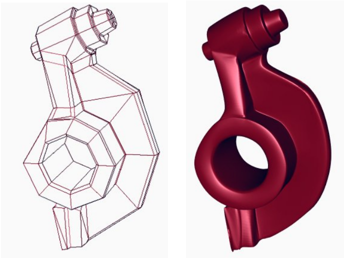
Interpolatory subdivision in CAD systems

- Interpolation is highly desirable because of its intuitive link with the starting mesh.



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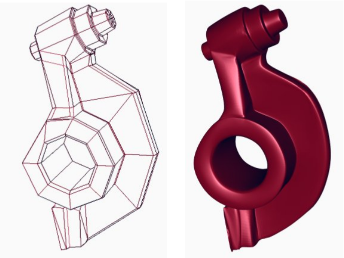

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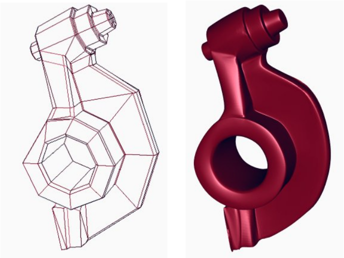

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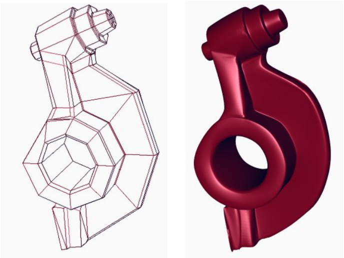

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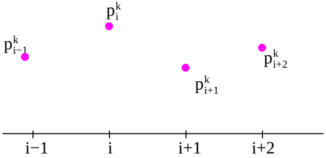

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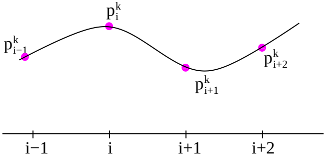

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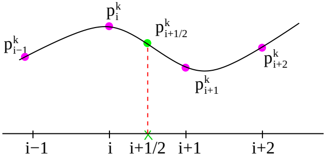

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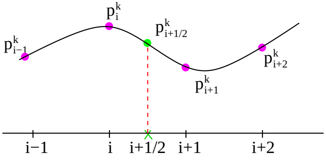

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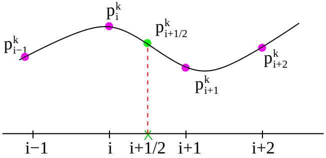

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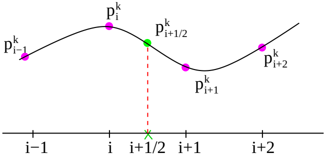

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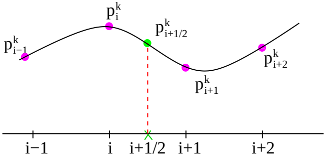

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- The curve scheme should be:
  - local
  - non-uniform: to follow closely the behavior of the starting polyline and modify the shape in a local way
  - non-stationary: to reproduce conic sections.

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Interpolatory curve subdivision with spline quality
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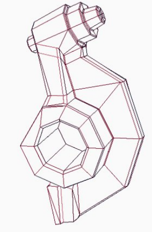

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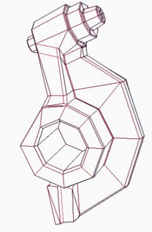

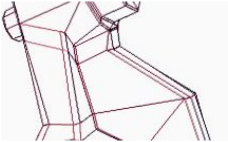

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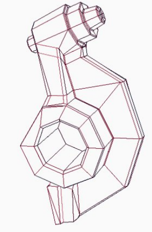

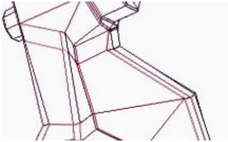

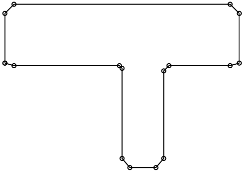

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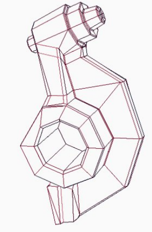
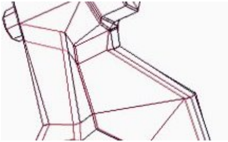
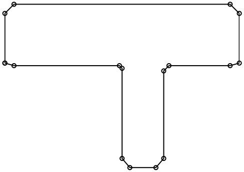
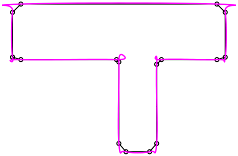






Splines naturally support non-uniform parameterizations!



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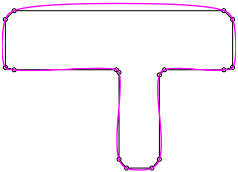

NON-UNIFORM SPLINE INTERPOLATION
centripetal parameterization

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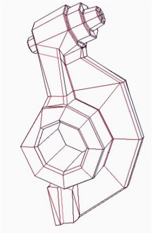

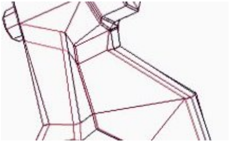

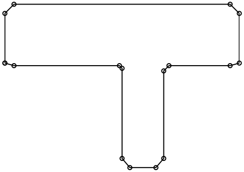

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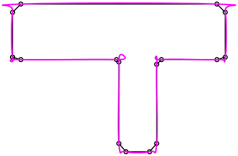

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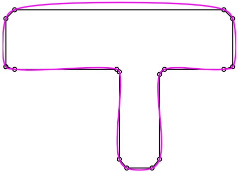

NON-UNIFORM SPLINE INTERPOLATION
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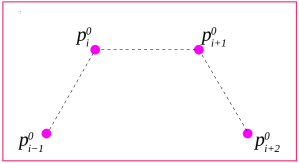
➥ Non-uniform parameters improve shape quality and alleviate undesired ondulations (Floater '08, Dyn et al. '07,...).

Spline based 4-pt subdivision



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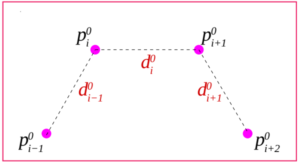

- initial parameterization

$$d_i^0 = \|p_{i+1}^0 - p_i^0\|_2^{\frac{1}{2}}$$ **centripetal**

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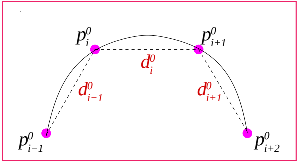

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  $\phi(x)$ non-uniform local interpolatory cardinal spline basis (Chui et. al. '96)

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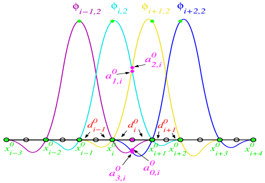
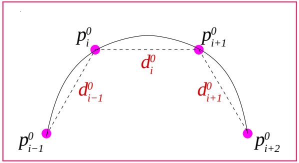


PROPERTIES of $\phi(x)$:

- arbitrary knots
- compact support $[x_{i-2}, x_{i+2}]$
- C^1 continuity
- polynomial reproduction up to deg. 2



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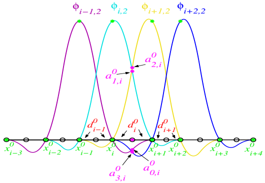

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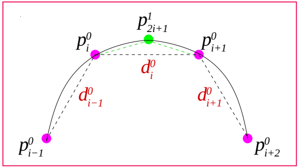

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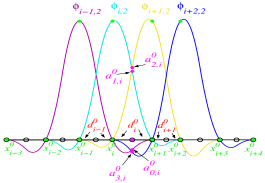

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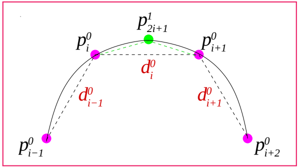

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  $\phi(x)$ non-uniform local interpolatory cardinal spline basis (Chui et. al. '96)

$$p^{k+1}_{2i+1} = p^k_{i+\frac{1}{2}} = f\left(\frac{x}{2}\right) \ldots$$

➡ **REFINEMENT EQUATIONS** at step k

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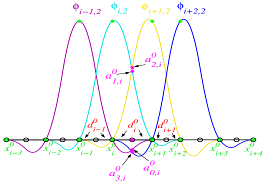
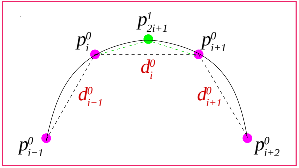
Spline based 4-pt subdivision



PROPERTIES of $\phi(x)$:

- arbitrary knots
- compact support $[x_{i-2}, x_{i+2}]$
- C^1 continuity
- polynomial reproduction up to deg. 2



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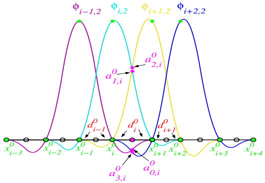

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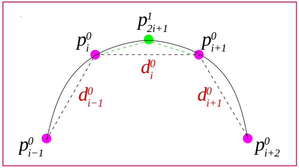

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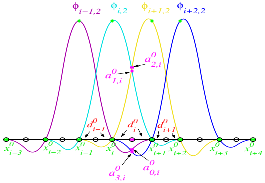
$a^k_{0,i}, \ldots, a^k_{3,i}$ coincide with the values of the deg-2 basis functions $\phi_{i-1,2}, \ldots, \phi_{i+2,2}$ defined on the refined knot-partition with $d^0_i/2^k$-length intervals

Spline based 4-pt subdivision



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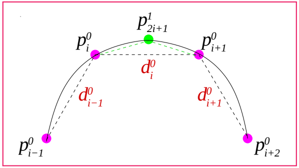

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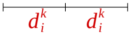

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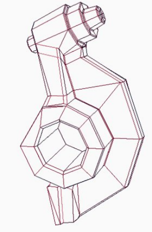
Subdivision surfaces & CAD

- interpolation
- spline quality (non-uniform parameterization)
- smoothness (C^k, $k \geq 1$)
- polynomial reproduction
- exact reproduction of circular features
- exact evaluation at arbitrary parameter values

Subdivision surfaces & CAD



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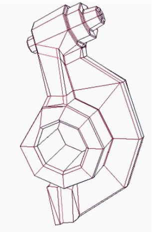

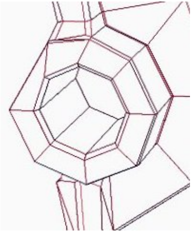

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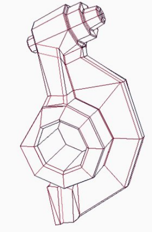
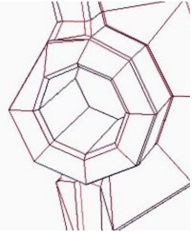
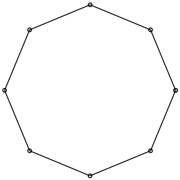


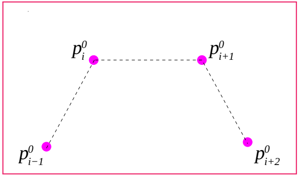




➥ Whenever the initial points of a section curve lie on a circular arc, this arc must be automatically detected and reproduced.

4-pt subdivision exact for exponentials

(B., Casciola, Romani - CAGD ’07)



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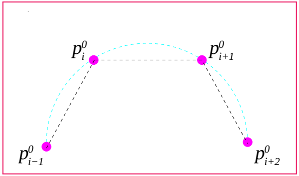
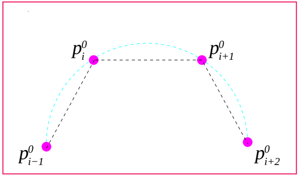


4-pt subdivision exact for exponentials
(B., Casciola, Romani - CAGD '07)

- $f(x) = c_0 + c_1\, x + c_2\, e^{isx} + c_3\, e^{-isx},\ s > 0$



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- $f(x) = c_0 + c_1\, x + c_2\, e^{isx} + c_3\, e^{-isx}$, $s > 0$
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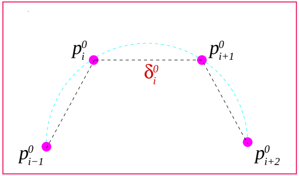

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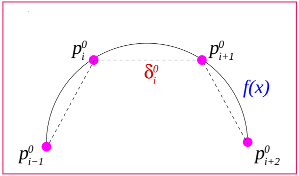

- **REMARK:** if the initial points are placed at equally-spaced parameter values on a circle, $f(x)$ reproduces the circle passing through them

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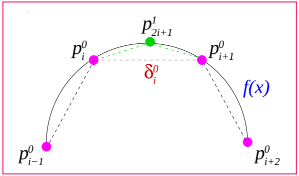

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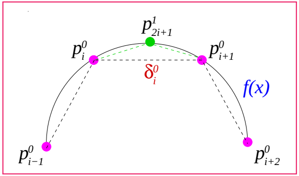

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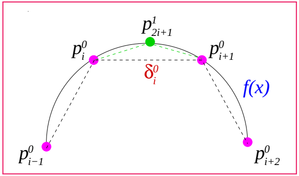

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4-pt subdivision exact for exponentials

(B., Casciola, Romani - CAGD '07)

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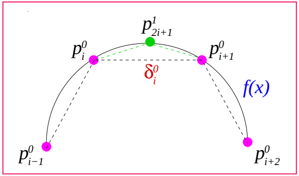

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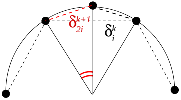
- **PARAMETERS UPDATING**

$$\delta_{2i}^{k+1} = \delta_{2i+1}^{k+1} = \sqrt{\frac{1+\delta_i^k}{2}} \quad \text{(half "arc length")}$$

parameters are not recomputed at each step, but they are updated automatically!



The NULI4^{++} curve scheme

- **4-pt REFINEMENT EQUATIONS**

$$p_{2i}^{k+1} = p_i^k$$
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- **REFINEMENT COEFFICIENTS**

SPLINE-BASED 4-pt

$$a_{0,i}^k = -\frac{1}{8\lambda_i^k(\lambda_i^k+1)}$$
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CIRCLE-PRESERVING 4-pt

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- **4-pt REFINEMENT EQUATIONS**

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$$\lambda_i = \mu_i$$

$$=$$

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- **REFINEMENT COEFFICIENTS**

SPLINE-BASED 4-pt

$$a_{0,i}^k = -\frac{1}{8\lambda_i^k(\lambda_i^k+1)}$$
$$a_{1,i}^k = \frac{1+3\lambda_i^k+\mu_i^k+4\lambda_i^k\mu_i^k}{\lambda_i^k(\mu_i^k+1)}$$
$$a_{2,i}^k = \frac{1+3\mu_i^k+\lambda_i^k+4\lambda_i^k\mu_i^k}{\mu_i^k(\lambda_i^k+1)}$$
$$a_{3,i}^k = -\frac{1}{8\mu_i^k(1+\mu_i^k)}$$

$$\lambda_i = \mu_i$$

$$=$$

CIRCLE-PRESERVING 4-pt

$$a_{0,i}^k = -\frac{1}{8\delta_i^k(1+\delta_i^k)}$$
$$a_{1,i}^k = \frac{(2\delta_i^k+1)^2}{8\delta_i^k(1+\delta_i^k)}$$
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- Define $\lambda_i^k = \frac{d_{i-1}^k}{d_i^k}\,\delta_i^k$ and $\mu_i^k = \frac{d_{i+1}^k}{d_i^k}\,\delta_i^k$
- Then the two subdivision schemes are represented by the same set of coefficients

The NULI4^{++} curve scheme

- **4-pt REFINEMENT EQUATIONS**

$$p_{2i}^{k+1} = p_i^k$$
$$p_{2i+1}^{k+1} = a_{0,i}^k\, p_{i-1}^k + a_{1,i}^k\, p_i^k + a_{2,i}^k\, p_{i+1}^k + a_{3,i}^k\, p_{i+2}^k$$

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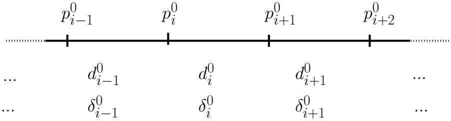
➥ **NULI4^{++} subdivision**

(Non-Uniform Local Interpolatory 4-pt Subdivision)

Initial parameters setting

- Two sets of parameters:



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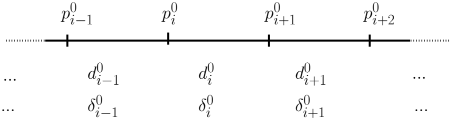

- **step1:** $\forall i$, compute $d^0_i = \|p^0_{i+1} - p^0_i\|_2^{\frac{1}{2}}$

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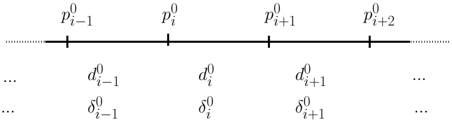


- **step1:** $\forall i$, compute $d^0_i = \|p^0_{i+1} - p^0_i\|_2^{\frac{1}{2}}$

- **step2:** $\forall i$, compute δ^0_i as follows

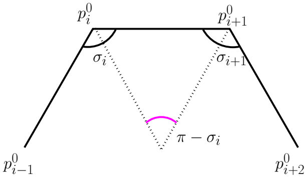
set $\delta^0_i = 1$

if $d^0_{i-1} = d^0_i = d^0_{i+1}$

compute σ_i and σ_{i+1}

if $\sigma_i = \sigma_{i+1}$

$\delta^0_i = \cos\left(\frac{\pi - \sigma_i}{2}\right)$



Parameters updating and refinement

For all $k \geq 1$

- **PARAMETERS UPDATING**

if $\delta_i^k \neq 1$ (i.e. $d_{i-1}^k = d_i^k = d_{i+1}^k$)

$$d_{2i}^{k+1} = d_{2i+1}^{k+1} = d_i^k$$

$$\delta_{2i}^{k+1} = \delta_{2i+1}^{k+1} = \sqrt{\frac{1+\delta_i^k}{2}}$$

else $(\delta_i^k = 1)$

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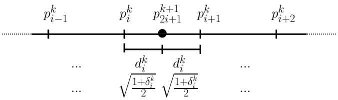

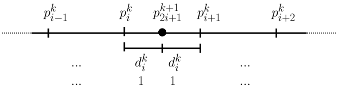

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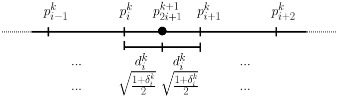

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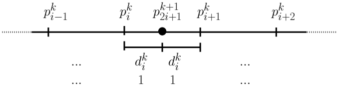

- **COEFFICIENTS COMPUTATION**

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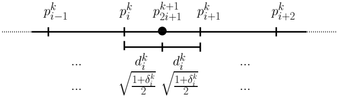

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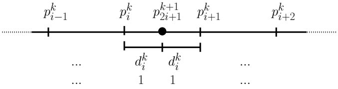

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Properties of the NULI4^{++} subdivision scheme

- non-uniform parameterization
- local support [-3,3]
- C^1 smoothness
- polynomials reproduction:
  - up to deg-2 starting from *non-equispaced* samples
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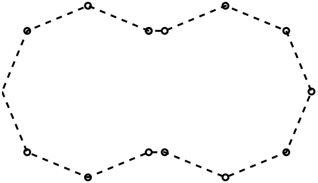

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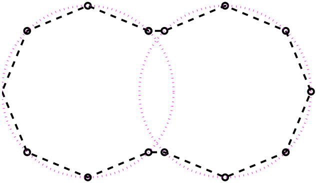

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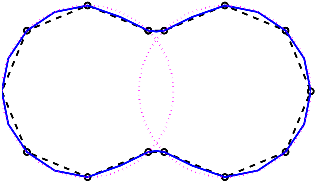

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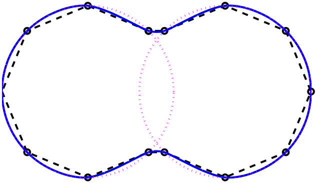

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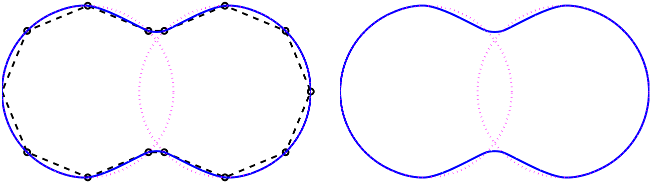
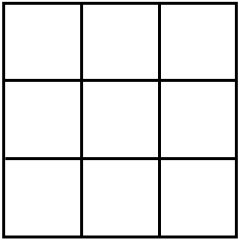


From curve subdivision to surface subdivision

- Quadrilateral meshes



From curve subdivision to surface subdivision

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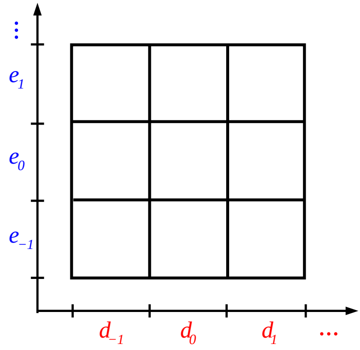

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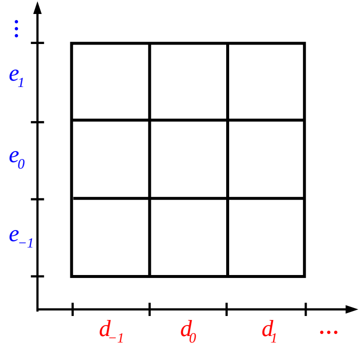

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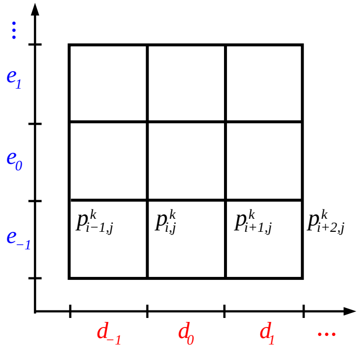

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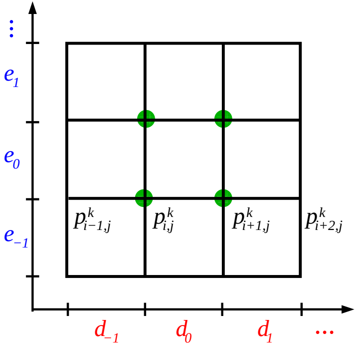

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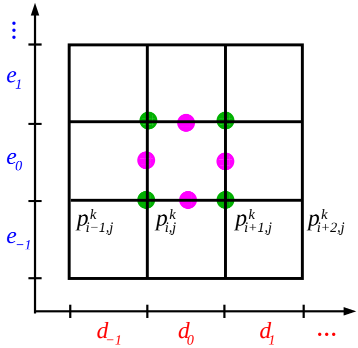

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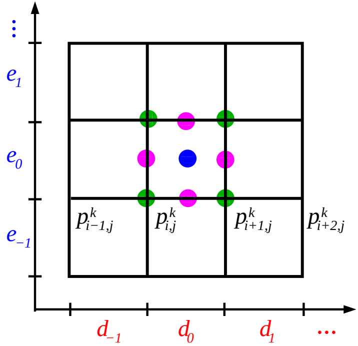

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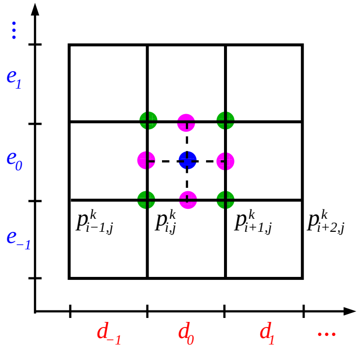

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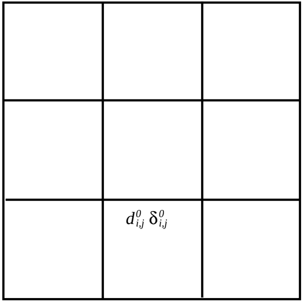
NULISS^{++} refinement algorithm

- A couple of parameters

$$\boldsymbol{d_j} = (d^0_{i,j}, \delta^0_{i,j})$$

is associated to each edge of the initial mesh



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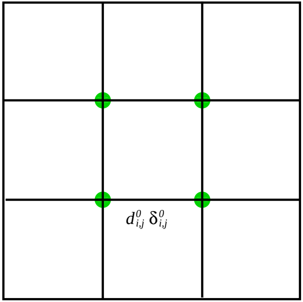

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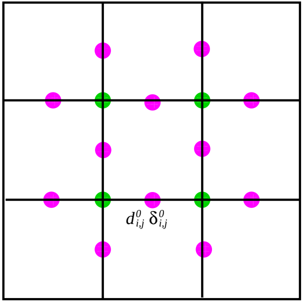

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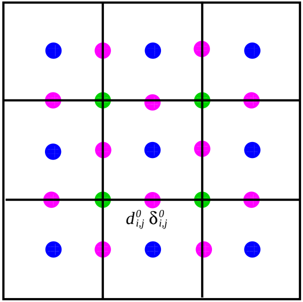

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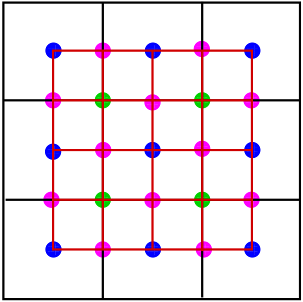

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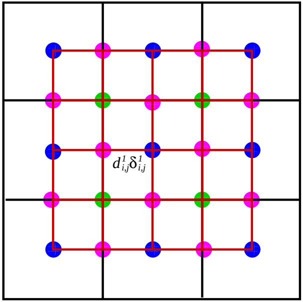

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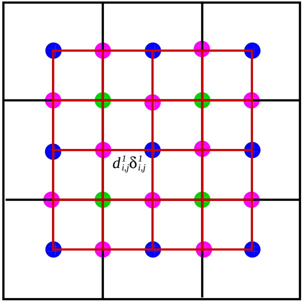

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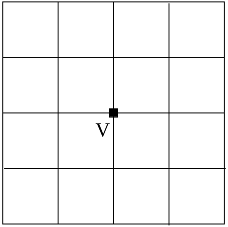
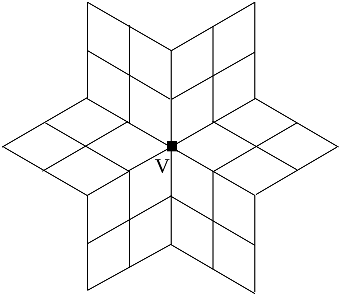
Taxonomy and parameters setting

- **REGULAR VERTICES ($n = 4$):** the refinement equations are derived by generalizing the tensor prouct of the curve scheme
- **EXTRAORDINARY VERTICES ($n \neq 4$):** the refinement equations are generalized, s. t. they reduce to the regular case when $n = 4$.





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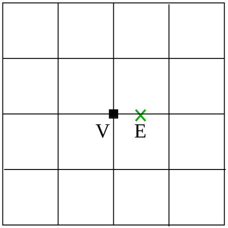

Parameters to compute the refinement coefficients:

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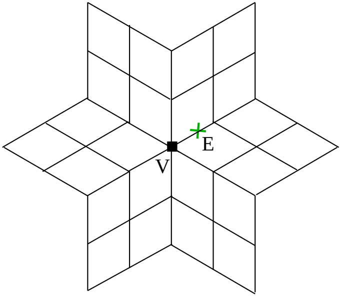

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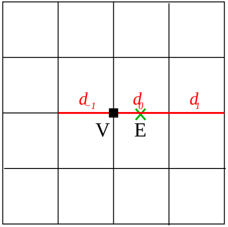

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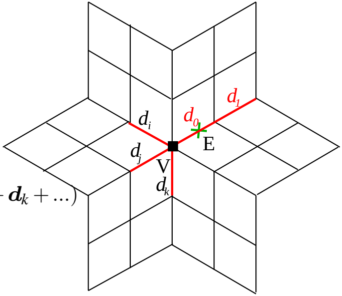

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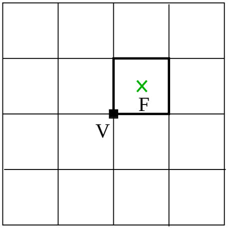

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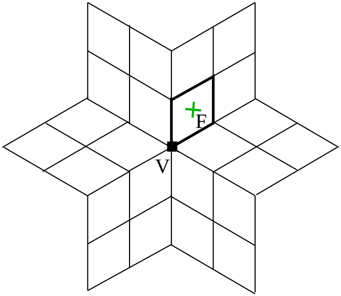

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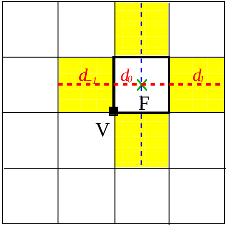

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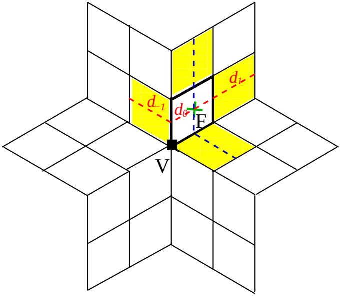
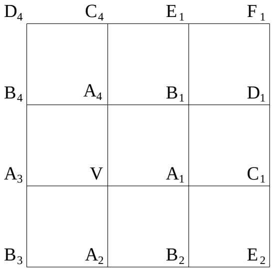


Regular vertices



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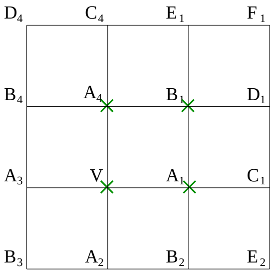

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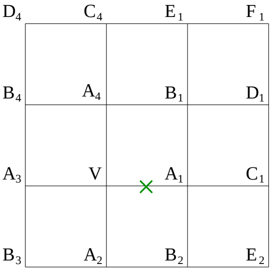

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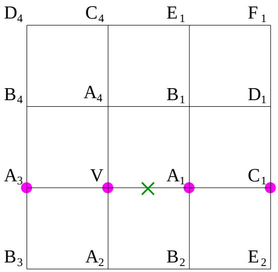

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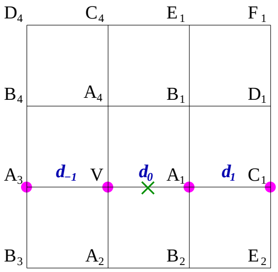

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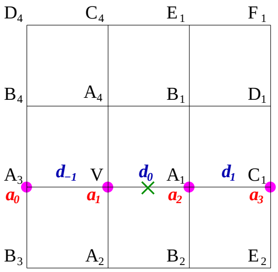
$$\mathbf{E} = a_0 A_3 + a_1 V + a_2 A_1 + a_3 C_1$$



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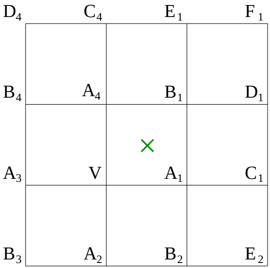

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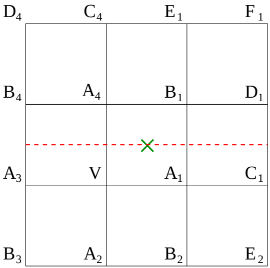

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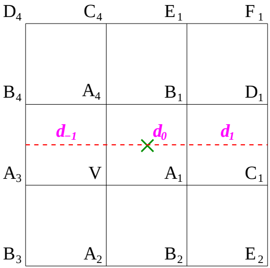

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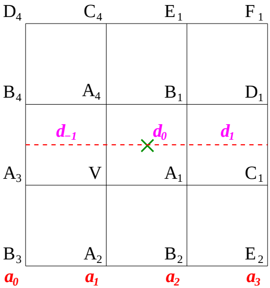

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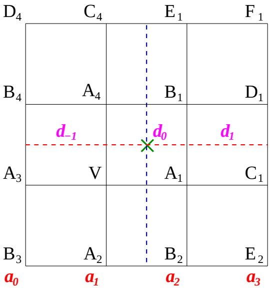

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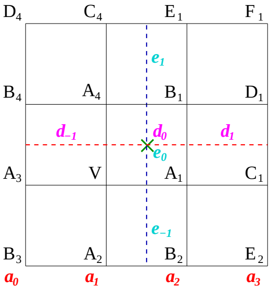

Regular vertices

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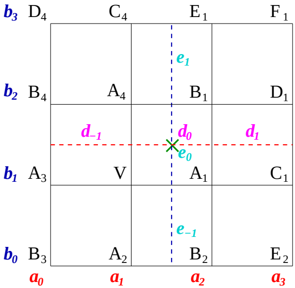
$$\mathbf{E} = a_0 A_3 + a_1 V + a_2 A_1 + a_3 C_1$$

- **FACE POINTS**

$$\begin{aligned}\mathbf{F} &= b_0(a_0 B_3 + a_1 A_2 + a_2 B_2 + a_3 E_2)\\ &+ b_1(a_0 A_3 + a_1 V + a_2 A_1 + a_3 C_1)\\ &+ b_2(a_0 B_4 + a_1 A_4 + a_2 B_1 + a_3 D_1)\\ &+ b_3(a_0 D_4 + a_1 C_4 + a_2 E_1 + a_3 F_1)\end{aligned}$$



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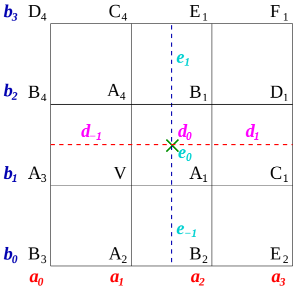
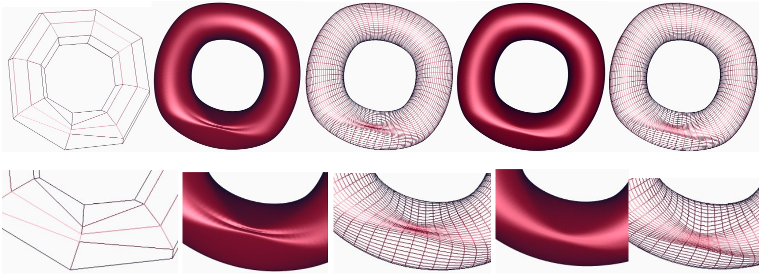


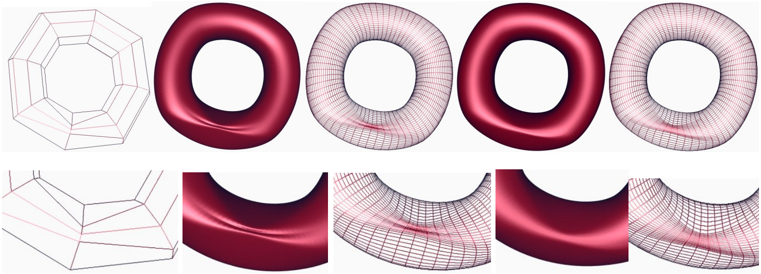
- When the parameters d_i are set compatibly with the tensor-product structure, the surface scheme NULISS^{++} coincides with the tensor-product of the NULI4^{++} curve scheme.

Regular case, non tensor-product

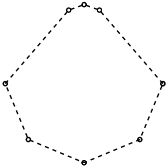


INITIAL MESH **SPLINE SURFACE** **NULISS^{++}**

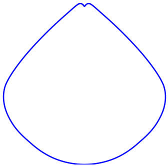
Regular case, non tensor-product



INITIAL MESH

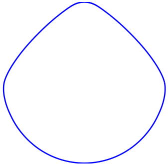


SPLINE SURFACE



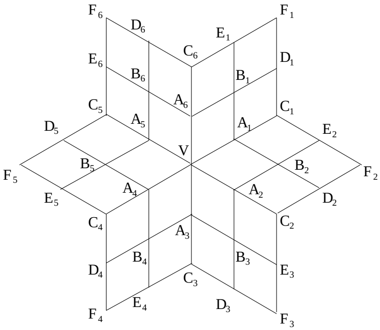
Each section curve is parameterized by the mean of the parameterizations in the corresponding direction

NULISS^{++}



Each section curve maintains its own (centripetal) parameterization

Extraordinary vertices, $n \geqslant 5$



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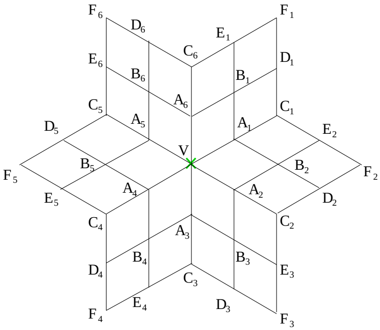

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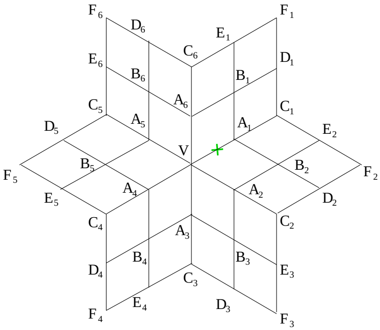
- **EDGE POINTS**

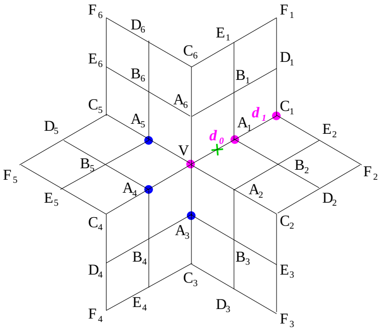


Extraordinary vertices, $n \geqslant 5$

- **VERTEX POINTS** are interpolated
- **EDGE POINTS**

$$d_{-1} = ?$$

$$\mathbf{E} = ? + ?\, V + a_2 A_1 + a_3 C_1$$



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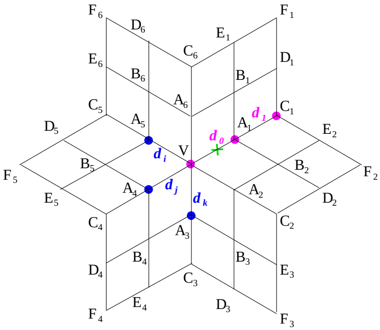
- **VERTEX POINTS** are interpolated
- **EDGE POINTS**

$$d_{-1} = \frac{1}{n-3}(d_i + d_j + d_k)$$

$$\mathbf{E} = a_0 \sum_{i=3}^{n-1} A_i + a_1 V + a_2 A_1 + a_3 C_1$$



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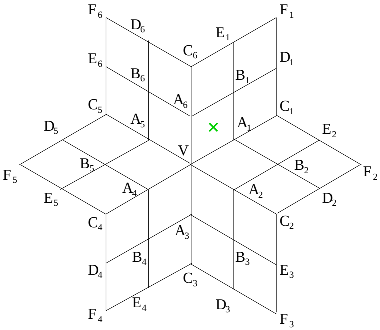

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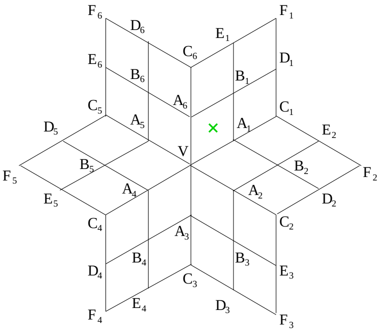

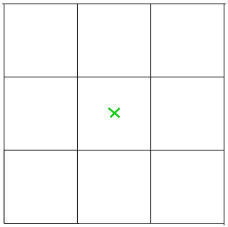

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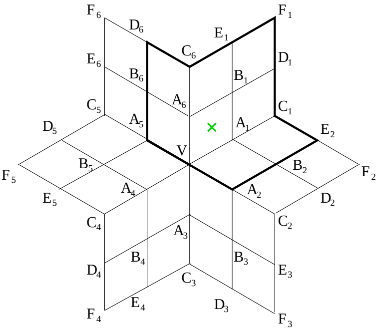

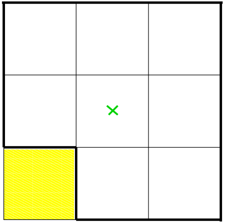

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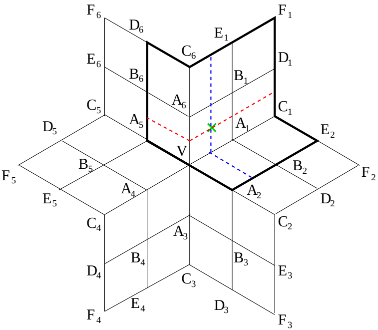
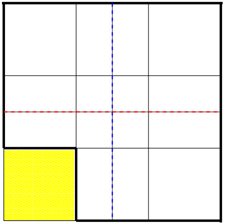
  parameters $\boldsymbol{d}_i$ are computed by averaging edge parameters in the two directions





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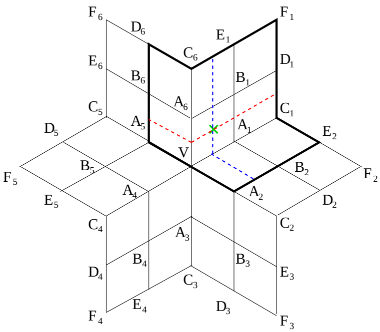

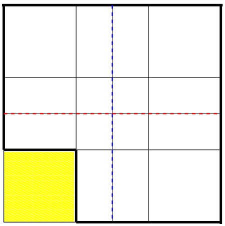

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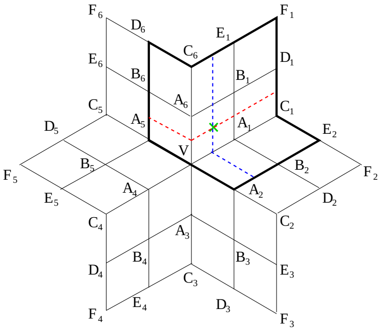
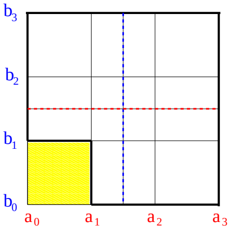
$$\begin{aligned}\mathbf{F} = \; & b_0(a_0 ? + a_1 ? + a_2 B_2 + a_3 E_2) \\ + \; & b_1(a_0 ? + a_1 V + a_2 A_1 + a_3 C_1) \\ + \; & b_2(a_0 B_6 + a_1 A_6 + a_2 B_1 + a_3 D_1) \\ + \; & b_3(a_0 D_6 + a_1 C_6 + a_2 E_1 + a_3 F_1)\end{aligned}$$





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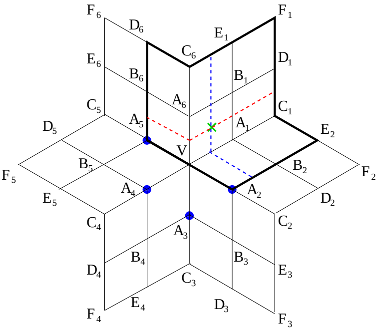

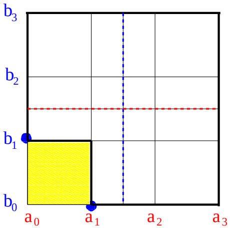

Extraordinary vertices, $n \geqslant 5$

- **VERTEX POINTS** are interpolated
- **EDGE POINTS**

$$\boldsymbol{d}_{-1} = \frac{1}{n-3}(\boldsymbol{d}_i + \boldsymbol{d}_j + \boldsymbol{d}_k)$$

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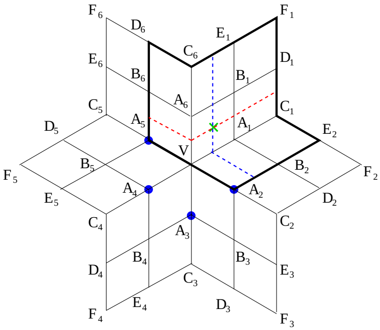

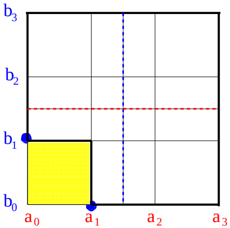

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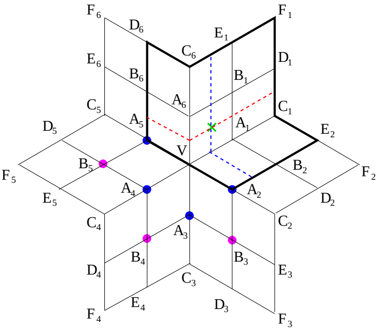

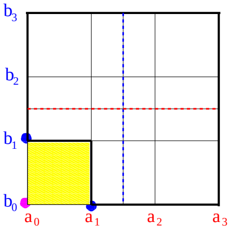

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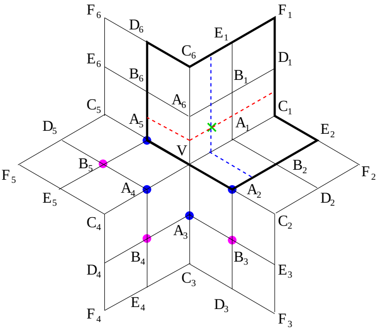

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$$\mathbf{F} = b_0\left(a_0 \frac{1}{n-3}\sum_{i=3}^{n-1} B_i + a_1 \frac{1}{n-3}\sum_{i=2}^{n-2} A_i + a_2 B_2 + a_3 E_2\right)$$
$$+ b_1\left(a_0 \frac{1}{n-3}\sum_{i=3}^{n-1} A_i + a_1 V + a_2 A_1 + a_3 C_1\right)$$
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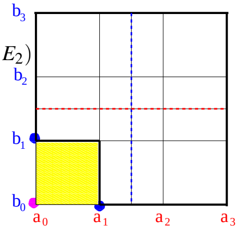

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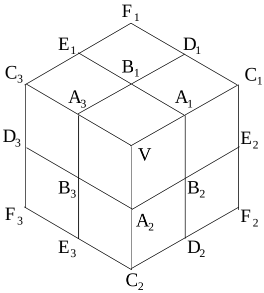

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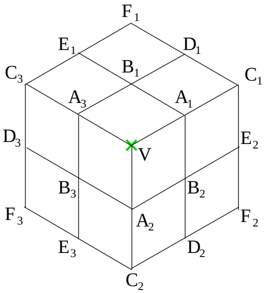

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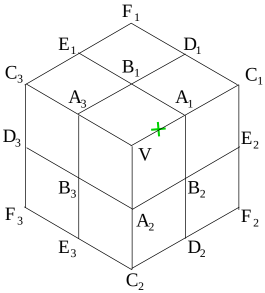

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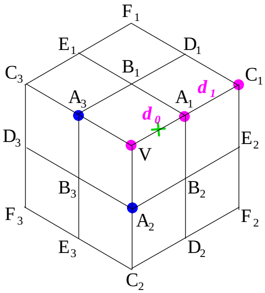

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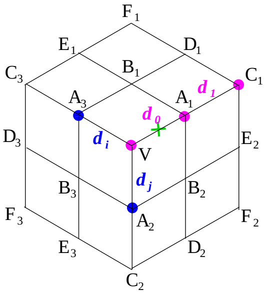

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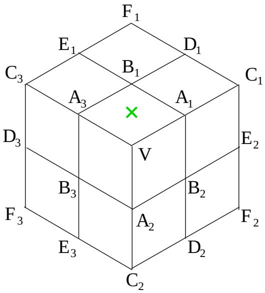

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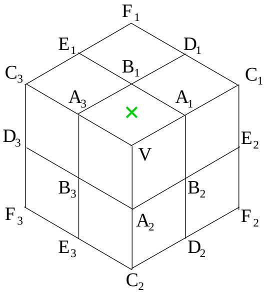

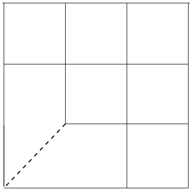

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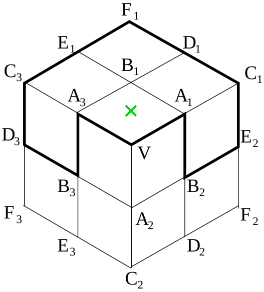

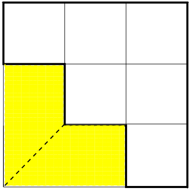

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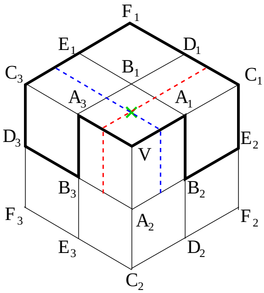

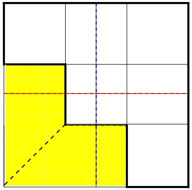

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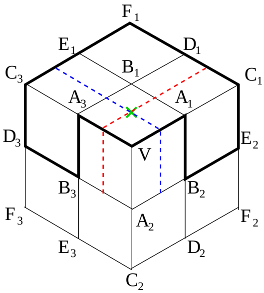

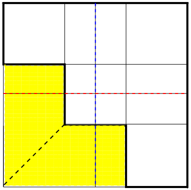

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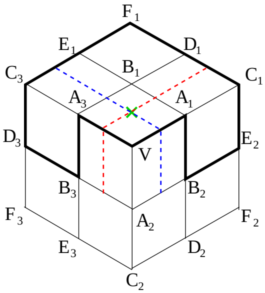

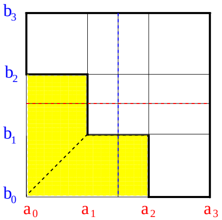

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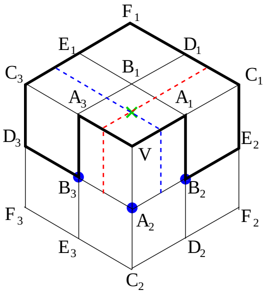

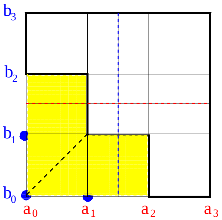

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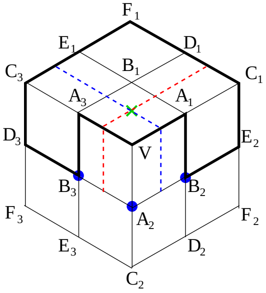

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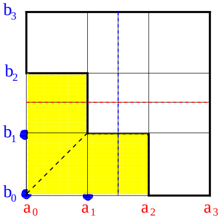

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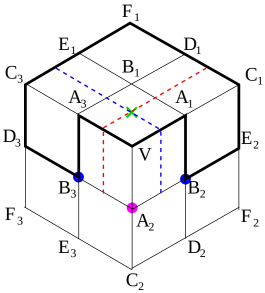

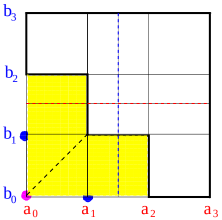

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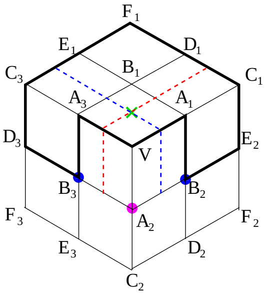

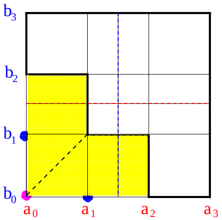

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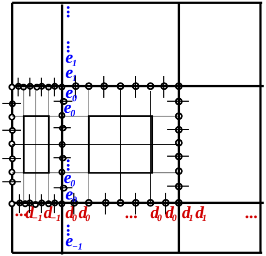

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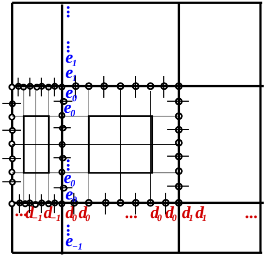
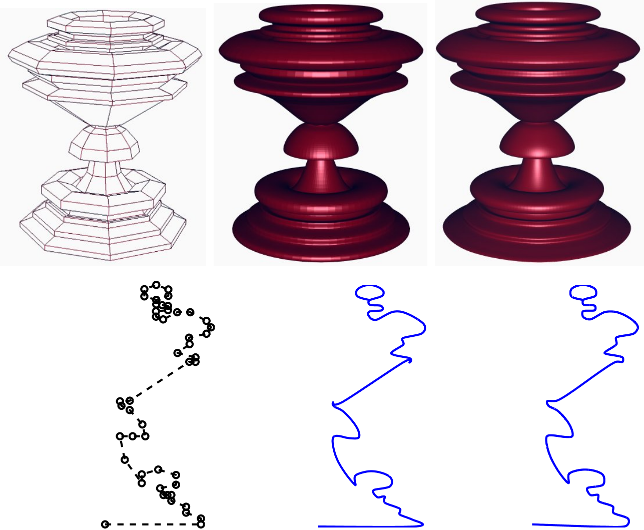


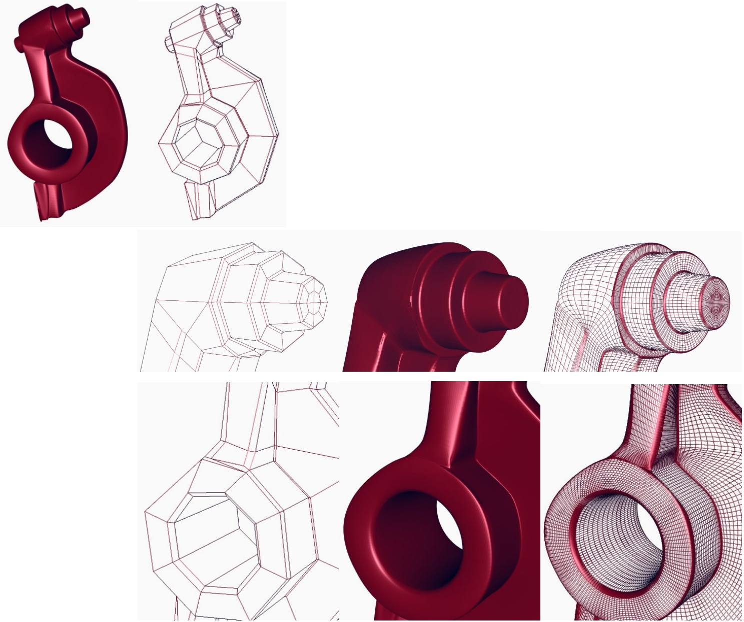
- after a few subdivision steps, evaluation in quadrilateral, tensor-product regions can be performed by the corresponding spline basis

Examples: uniform and non-uniform subdivision



initial mesh/polyline — uniform subdivision — non-uniform subdivision

Examples: extraordinary vertices and circular features



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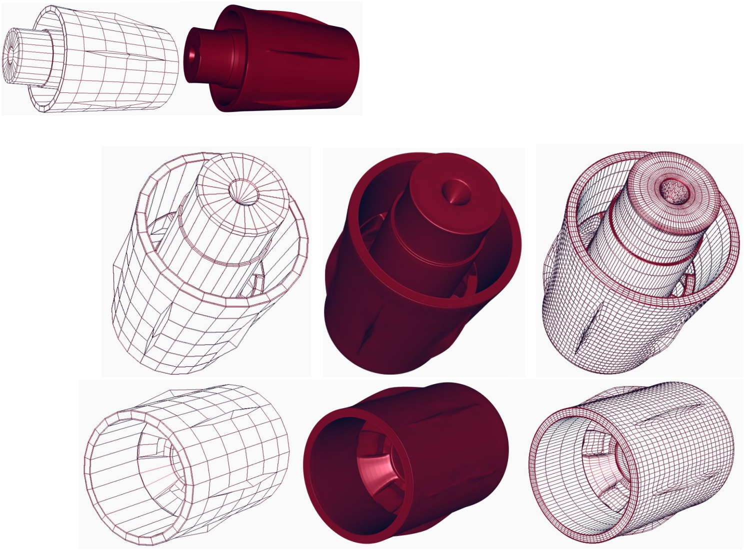

Conclusions and future work

Subdivision surfaces & CAD

- ✔ interpolation
- ✔ spline quality (non-uniform parameterization)
- ✔ exact reproduction of circular features
- ✔ polynomial reproduction
- ✔ C^1 smoothness in tensor-product regular regions
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Future work:

- smoothness in non tensor-product regions
- evaluation at arbitrary parameter values