Non-uniform interpolatory subdivision designed from splines

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THE MOST EFFECTIVE METHODS FOR CURVE AND SURFACE DESIGN generate a shape that

- passes through all the vertices of a given control net
- faithfully mimics its behaviour (no undesired undulations)
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- \textit{spline-based methods}:
  approximating and interpolating
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generate a shape that

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**Used in CAGD:**

- *spline-based methods:*
  - approximating and interpolating
- *subdivision methods:*
  - approximating and spline-reproducing
THE MOST EFFECTIVE METHODS FOR CURVE AND SURFACE DESIGN

generate a shape that

- passes through all the vertices of a given control net
- faithfully mimics its behaviour (no undesired undulations)

**Used in CAGD:**
- *spline-based methods:*
  - approximating and interpolating
- *subdivision methods:*
  - approximating and spline-reproducing

**Proposed HERE:**
*interpolatory subdivision methods*
designed from a class of
non-uniform local interpolatory cardinal splines (NULICS)
Non-uniform local interpolatory cardinal splines (NULICS)

W. Dahmen, T.N.T Goodman, C.A. Micchelli (1988): locally-supported fundamental splines leading to highly accurate local interpolation methods

- require the solution of small linear systems (depend on the spline degree, but not on the number of interpolation points)
Non-uniform local interpolatory cardinal splines (NULICS)

  - require the solution of small linear systems (depend on the spline degree, but not on the number of interpolation points)

Motivations

Non-uniform local interpolatory cardinal splines (NULICS)

The NULI 4-pt scheme

Application examples

Conclusions

Non-uniform local interpolatory cardinal splines (NULICS)

  locally-supported fundamental splines leading to highly accurate local
  interpolation methods

  ➡ require the solution of small linear systems (depend on the spline
  degree, but not on the number of interpolation points)

  introduction of explicitly represented local interpolatory cardinal splines
  and their applications (uniform case)

  construction of general-degree NULICS and explicit formulation of their
  coefficients in terms of the data points

  ➡ no solution of linear systems is required
Main features of the degree-$n$ local interpolatory cardinal spline basis:

- arbitrary knots $\{x_j\}_{j \in \mathbb{Z}}$
- compact support $[x_j-n, x_j+n]$
- $C^{n-1}$ continuity
- polynomials reproduction up to degree $n$, starting from arbitrarily non-equispaced samples
Main features of the degree-$n$ local interpolatory cardinal spline basis:

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- compact support $[x_j-n, x_j+n]$
- $C^{n-1}$ continuity
- polynomials reproduction up to degree $n$, starting from arbitrarily non-equispaced samples

No application example of the very general non-uniform case (probably due to its involved and complex representation)
NULICS: effectiveness of the local interpolatory method

Quadratic cardinal spline interpolation

UNIFORM KNOTS

NON-UNIFORM KNOTS
NULICS: effectiveness of the local interpolatory method

Quadratic cardinal spline interpolation

UNIFORM KNOTS

NON-UNIFORM KNOTS

CENTRIPETAL PARAMETERIZATION

▷ M. Floater (2008): advantages of the centripetal parameterization in spline interpolation

▷ N. Dyn, M. Floater, K. Hormann (2007): introduction of non-uniform parameters in interpolatory 4-point subdivision
The NULICS is not refirable!
The NULICS is not refinable!

- There is no interpolatory subdivision scheme converging to the NULICS basis in the limit.
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- There is no interpolatory subdivision scheme converging to the NULICS basis in the limit.

But

- We can design NULI insertion rules by recursively upsampling the NULICS basis at its mid-knots.
The NULICS is not refinable!

- There is no interpolatory subdivision scheme converging to the NULICS basis in the limit.

But

- We can design NULI insertion rules by recursively upsampling the NULICS basis at its mid-knots.

- In general: degree-$n$ NULICS $\Rightarrow$ 2$n$-point NULI refinement equations
The NULICS is not refinable!

- There is no interpolatory subdivision scheme converging to the NULICS basis in the limit.

But

- We can design NULL insertion rules by recursively upsampling the NULICS basis at its mid-knots.

- In general: degree-\(n\) NULICS \(\Rightarrow\) \(2n\)-point NULL refinement equations

- In this talk: quadratic NULICS \(\Rightarrow\) NULL 4-point subdivision
The deg-2 NULICS basis

**Step 1: define the knot-partition**

\[ x_{j-2} \quad x_{j-1} \quad x_j \quad x_{j+1} \quad x_{j+2} \]

\[ x_{j+h} \quad (h = -2, \ldots, 2) \quad \text{arbitrary break points (parameters for points to be interpolated)} \]
The deg-2 NULICS basis

**Step 1: define the knot-partition**

\[
\begin{array}{llllllll}
  x_{j-2} & t_{j-4} & x_{j-1} & t_{j-3} & x_j & t_{j-2} & t_j & t_{j+1} & x_{j+1} & t_{j+2} & t_{j+3} & t_{j+4} & x_{j+2} \\
\end{array}
\]

\[t_{j+2h} \equiv x_{j+h} \quad (h = -2, \ldots, 2) \quad \text{knots for interpolation points}\]

\[t_{j+2h+1} = \frac{x_j + x_{j+h} + 1}{2} \quad (h = -2, \ldots, 1) \quad \text{intermediate knots}\]

\} \quad \text{complete knot sequence}
The deg-2 NULICS basis

**Step 1:** *define the knot-partition*

\[
\begin{align*}
  x_{j-2} & \quad t_{j-4} & \quad x_{j-1} & \quad t_{j-3} & \quad x_j & \quad t_{j-2} & \quad t_{j-1} & \quad t_j & \quad t_{j+1} & \quad x_{j+1} & \quad t_{j+2} & \quad t_{j+3} & \quad t_{j+4} \\
  t_{j-4} & \quad d_{j-2} & \quad t_{j-3} & \quad d_{j-2} & \quad t_{j-2} & \quad d_{j-1} & \quad t_{j-1} & \quad d_{j-1} & \quad t_j & \quad d_j & \quad t_{j+1} & \quad d_j & \quad t_{j+2} & \quad d_{j+1} & \quad t_{j+3} & \quad d_{j+1} & \quad t_{j+4}
\end{align*}
\]

\[t_{j+2h} \equiv x_{j+h} \quad (h = -2, \ldots, 2)\quad \text{knots for interpolation points}\]
\[t_{j+2h+1} = \frac{x_{j+h} + x_{j+h+1}}{2} \quad (h = -2, \ldots, 1)\quad \text{intermediate knots}\]
\[d_{j+h} \quad (h = -2, \ldots, 1)\quad \text{knot intervals}\]
The deg-2 NULICS basis

**Step 1:** define the knot-partition

\[
\begin{align*}
\begin{array}{cccccccc}
x_{j-2} & t_{j-4} & \cdots & t_{j-3} & t_{j-2} & \cdots & t_{j-1} & t_j & t_{j+1} & \cdots & x_{j+1} & t_{j+2} & \cdots & t_{j+3} & t_{j+4} \\
d_{j-2} & & \cdots & & d_{j-2} & & \cdots & d_{j-1} & d_j & \cdots & d_j & d_j+1 & \cdots & d_j+1
\end{array}
\end{align*}
\]

\[t_{j+2h} \equiv x_{j+h} \quad (h = -2, \ldots, 2) \quad \text{knots for interpolation points}\]

\[t_{j+2h+1} = \frac{x_{j+h} + x_{j+h+1}}{2} \quad (h = -2, \ldots, 1) \quad \text{intermediate knots}\]

\[d_{j+h} \quad (h = -2, \ldots, 1) \quad \text{knot intervals}\]

**Step 2:** represent the deg-2 NULICS basis centered at \(t_j\) in terms of B-spline basis functions

\[
\phi_{j,2}(x) = \sum_{i=0}^{5} b_i N_{i+j-4,2}(x)
\]

where

\[
\begin{align*}
b_0 &= -\frac{(d_{j-2})^2}{4d_{j-1}(d_{j-2}+d_{j-1})} \\
b_1 &= \frac{d_{j-2}}{4(d_{j-2}+d_{j-1})} \\
b_2 &= \frac{d_{j-1}+3d_j}{4d_j} \\
b_3 &= \frac{3d_{j-1}+d_j}{4d_j} \\
b_4 &= \frac{d_{j+1}}{4(d_j+d_{j+1})} \\
b_5 &= -\frac{(d_{j+1})^2}{4d_j(d_j+d_{j+1})}
\end{align*}
\]

and \(N_{i+j-4,2}(x)\) is the quadratic B-spline with support \([t_{i+j-4}, t_{i+j-1}]\).
The deg-2 NULICS basis

Step 1: \textit{define the knot-partition}

\[
\begin{align*}
x_j-2 & \quad t_j-4 & \quad x_j-1 & \quad t_j-3 & \quad x_j & \quad t_j-2 & \quad x_j & \quad t_j-1 & \quad x_j & \quad t_j & \quad x_j+1 & \quad t_j+1 & \quad x_j+2 & \quad t_j+2 & \quad x_j+3 & \quad t_j+3 & \quad t_j+4 \\
& \quad d_{j-2} & \quad & \quad & \quad & \quad d_{j-2} & \quad & \quad & \quad & \quad d_{j-1} & \quad & \quad & \quad & \quad d_{j-1} & \quad & \quad & \quad & \quad d_{j} & \quad & \quad & \quad & \quad d_{j} & \quad & \quad & \quad & \quad d_{j+1} & \quad & \quad & \quad & \quad d_{j+1} & \quad & \quad & \quad & \quad d_{j+1}
\end{align*}
\]

\[
t_{j+2h} \equiv x_{j+h} \quad (h = -2, \ldots, 2) \quad \text{knots for interpolation points}
\]

\[
t_{j+2h+1} = \frac{x_{j+h} + x_{j+h+1}}{2} \quad (h = -2, \ldots, 1) \quad \text{intermediate knots}
\]

\[
d_{j+h} \quad (h = -2, \ldots, 1) \quad \text{knot intervals}
\]

\[
\begin{align*}
\text{Step 2: } & \text{represent the deg-2 NULICS basis centered at } t_j \text{ in terms of B-spline basis functions} \\
\phi_{j,2}(x) &= \sum_{i=0}^{5} b_i N_{i+j-4,2}(x)
\end{align*}
\]

where

\[
\begin{align*}
b_0 &= -\frac{(d_{j-2})^2}{4d_{j-1}(d_{j-2}+d_{j-1})} \\
b_1 &= \frac{d_{j-2}}{4(d_{j-2}+d_{j-1})} \\
b_2 &= \frac{d_{j-1}+3d_{j}}{4d_j} \\
b_3 &= \frac{3d_{j-1}+d_{j}}{4d_{j-1}} \\
b_4 &= \frac{d_{j+1}}{4(d_{j}+d_{j+1})} \\
b_5 &= -\frac{(d_{j+1})^2}{4d_j(d_{j}+d_{j+1})}
\end{align*}
\]

and \( N_{i+j-4,2}(x) \) is the quadratic B-spline with support \([t_{i+j-4}, t_{i+j-1}]\).
The deg-2 NULICS basis

\[ \phi_{j,2}(x_j) = 1, \quad \phi_{j,2}(x_{j+h}) = 0 \quad \forall h \neq 0 \]
Towards a NULI 4-pt scheme

- Let $p_{j+h}^0$ (h=-1,0,1,2) be the quadruple of starting points
Towards a NULI 4-pt scheme

- Let \( p_{j+h}^0 \) \((h=-1,0,1,2)\) be the quadruple of starting points

- Let \( x_{j+h}^0 \) \((h=-1,0,1,2)\) be the *centripetal parameter values* of \( p_{j+h}^0 \)
Towards a NULI 4-pt scheme

- Let $p_{j+h}^0$ (h=-1,0,1,2) be the quadruple of starting points
- Let $x_{j+h}^0$ (h=-1,0,1,2) be the centripetal parameter values of $p_{j+h}^0$
- Compute $a_{j+h}^0 = \frac{x_{j+h+1}^0 - x_{j+h}^0}{2}$ (h=-1,0,1)
Towards a NULI 4-pt scheme

- Let $p_{j+h}^0$ (h=-1,0,1,2) be the quadruple of starting points
- Let $x_{j+h}^0$ (h=-1,0,1,2) be the centripetal parameter values of $p_{j+h}^0$
- Compute $d_{j+h}^0 = \frac{x_{j+h+1}^0 - x_{j+h}^0}{2}$ (h=-1,0,1)
- Let $\phi_{j+h,2}(x)$ be the NULICS basis centered at $x_{j+h}^0$ (h=-1,0,1,2)
Towards a NULI 4-pt scheme

- Let \( p_{j+h}^0 \) (h=-1,0,1,2) be the quadruple of starting points
- Let \( x_{j+h}^0 \) (h=-1,0,1,2) be the *centripetal parameter values* of \( p_{j+h}^0 \)
- Compute \( d_{j+h}^0 = \frac{x_{j+h+1}^0 - x_{j+h}^0}{2} \) (h=-1,0,1)
- Let \( \phi_{j+h,2}(x) \) be the NULICS basis centered at \( x_{j+h}^0 \) (h=-1,0,1,2)
- Compute \( a_{h+1,j}^0 = \phi_{j+h,2} \left( \frac{x_{j}^0 + x_{j+1}^0}{2} \right) \) (h=-1,0,1,2)
Towards a NULI 4-pt scheme

\[
\begin{align*}
  a_{0,j}^0 &= -\frac{(d_j^0)^2}{8a_{j-1}^0(d_{j-1}^0 + d_j^0)} \\
  a_{1,j}^0 &= \frac{(d_j^0)^2 + (3d_{j-1}^0 + d_{j+1}^0)d_j^0 + 4d_{j-1}^0 d_{j+1}^0}{8a_{j-1}^0(d_j^0 + d_{j+1}^0)} \\
  a_{2,j}^0 &= \frac{(d_j^0)^2 + (d_{j-1}^0 + 3d_{j+1}^0)d_j^0 + 4d_{j-1}^0 d_{j+1}^0}{8a_{j+1}^0(d_j^0 + d_{j+1}^0)} \\
  a_{3,j}^0 &= -\frac{(d_j^0)^2}{8a_{j+1}^0(d_j^0 + d_{j+1}^0)}
\end{align*}
\]
The NULI 4-pt scheme

- $d^0_j$ starting parameters

At each step $k \geq 0$

**REFINEMENT EQUATIONS**

\[
\begin{align*}
    p_{2j}^{k+1} &= p_j^k \\
    p_{2j+1}^{k+1} &= a_{0,j}^k p_j^{k-1} + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k
\end{align*}
\]

\[
\begin{align*}
    a_{0,j}^k &= -\frac{(d_j^k)^2}{8d_j^k(d_j^k-1+d_j^k)} \\
    a_{1,j}^k &= \frac{(d_j^k)^2 + (3d_j^k - 1 + d_j^k+1)d_j^k + 4d_j^k - 1}{8d_j^k-1(d_j^k+d_j^k+1)} \\
    a_{2,j}^k &= \frac{(d_j^k)^2 + (d_j^k-1+3d_j^k+1)d_j^k + 4d_j^k - 1}{8d_j^k+1(d_j^k-1+d_j^k)} \\
    a_{3,j}^k &= -\frac{(d_j^k)^2}{8d_j^k+1(d_j^k+d_j^k+1)}
\end{align*}
\]
The NULI 4-pt scheme

Starting parameters

At each step $k \geq 0$

\[
\begin{align*}
d_j^0 & \quad \text{starting parameters} \\
p_{2j}^{k+1} & = p_j^k \\
p_{2j+1}^{k+1} & = a_{0,j}^k p_{j-1}^k + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k
\end{align*}
\]

REFINEMENT EQUATIONS

\[
\begin{align*}
a_{0,j}^k & = - \frac{(d_j^k)^2}{8d_{j-1}^k(d_j^k - d_j^k + 1)} \\
a_{1,j}^k & = \frac{(d_j^k)^2 + (3d_j^k - 1 + d_j^k + 1)d_j^k + 4d_j^k - 1d_j^k + 1}{8d_{j-1}^k(d_j^k + d_j^k + 1)} \\
a_{2,j}^k & = \frac{(d_j^k)^2 + (d_j^k - 1 + 3d_j^k + 1)d_j^k + 4d_j^k - 1d_j^k + 1}{8d_{j+1}^k(d_j^k - 1 + d_j^k)} \\
a_{3,j}^k & = - \frac{(d_j^k)^2}{8d_{j+1}^k(d_j^k + d_j^k + 1)}
\end{align*}
\]
The NULI 4-pt scheme

$\triangleright \quad d_j^0$ starting parameters

At each step $k \geq 0$

**REFINEMENT EQUATIONS**

\[
\begin{align*}
p_{2j}^{k+1} &= p_j^k \\
p_{2j+1}^{k+1} &= a_{0,j}^k p_{j-1}^k + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k
\end{align*}
\]

\[
\begin{align*}
a_{0,j}^k &= - \frac{(d_j^k)^2}{8d_j^{k-1}(d_j^k - d_{j-1}) (d_j^k + d_{j+1})} \\
a_{1,j}^k &= \frac{(d_j^k)^2 + (3d_j^{k-1} + d_{j+1}) d_j^k + 4d_j^{k-1} d_{j+1}}{8d_j^{k-1}(d_j^k + d_{j+1})} \\
a_{2,j}^k &= \frac{(d_j^k)^2 + (d_j^{k-1} + 3d_j^{k+1}) d_j^k + 4d_j^{k-1} d_{j+1}}{8d_j^{k+1}(d_j^k - d_{j+1})} \\
a_{3,j}^k &= - \frac{(d_j^k)^2}{8d_j^{k+1}(d_j^k + d_{j+1})}
\end{align*}
\]
The NULI 4-pt scheme

\( d_j^0 \) starting parameters

At each step \( k \geq 0 \)

**REFINEMENT EQUATIONS**

\[
\begin{align*}
p_{2j}^{k+1} & = p_j^k \\
p_{2j+1}^{k+1} & = a_{0,j}^k p_{j-1}^k + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k
\end{align*}
\]

\[
\begin{align*}
a_{0,j}^k & = -\frac{(d_j^k)^2}{8d_j^{k-1}(d_j^k + d_j^{k+1})} \\
a_{1,j}^k & = \frac{(d_j^k)^2 + (3d_j^{k+1} - d_j^k + 1) d_j^k + 4d_j^{k-1}d_j^{k+1}}{8d_j^{k-1}(d_j^k + d_j^{k+1})} \\
a_{2,j}^k & = \frac{(d_j^k)^2 + (3d_j^{k-1} + 3d_j^{k+1}) d_j^k + 4d_j^{k-1}d_j^{k+1}}{8d_j^{k-1}(d_j^k + d_j^{k+1})} \\
a_{3,j}^k & = -\frac{(d_j^k)^2}{8d_j^{k+1}(d_j^k + d_j^{k+1})}
\end{align*}
\]
The NULI 4-pt scheme

At each step $k \geq 0$

REFINEMENT EQUATIONS

\[
p_{2j}^{k+1} = p_{j}^{k}
\]
\[
p_{2j+1}^{k+1} = a_{0,j}^{k} p_{j-1}^{k} + a_{1,j}^{k} p_{j}^{k} + a_{2,j}^{k} p_{j+1}^{k} + a_{3,j}^{k} p_{j+2}^{k}
\]

\[
a_{0,j}^{k} = - \frac{(d_{j}^{k})^2}{8d_{j-1}^{k}(d_{j-1}^{k} + d_{j}^{k})}
\]
\[
a_{1,j}^{k} = \frac{(d_{j}^{k})^2 + (3d_{j-1}^{k} + d_{j+1}^{k})d_{j}^{k} + 4d_{j-1}^{k}d_{j+1}^{k}}{8d_{j-1}^{k}(d_{j-1}^{k} + d_{j}^{k} + d_{j}^{k} + 1)}
\]
\[
a_{2,j}^{k} = \frac{(d_{j}^{k})^2 + (d_{j-1}^{k} + 3d_{j+1}^{k})d_{j}^{k} + 4d_{j-1}^{k}d_{j+1}^{k}}{8d_{j+1}^{k}(d_{j-1}^{k} + d_{j}^{k})}
\]
\[
a_{3,j}^{k} = - \frac{(d_{j}^{k})^2}{8d_{j+1}^{k}(d_{j}^{k} + d_{j}^{k} + 1)}
\]
The NULI 4-pt scheme

$\Rightarrow \quad d^0_j \text{ starting parameters}$

At each step $k \geq 0$

**REFINEMENT EQUATIONS**

$p_{2j}^{k+1} = p_j^k$

$p_{2j+1}^{k+1} = a_{0,j}^k p_{j-1}^k + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k$

Coefficients $a_{0,j}^k, \ldots, a_{3,j}^k$ coincide with the values of the deg-2 basis functions $\phi_{j-1,2}, \ldots, \phi_{j+2,2}$ defined on the refined knot-partition with $d_j^0 / 2^k$-length intervals, at the central knot $x_j^k + x_{j+1}^k / 2$.

\[
\begin{align*}
a_{0,j}^k &= -\frac{(d_j^k)^2}{8d_{j-1}^k (d_{j-1}^k + d_j^k)} \\
\frac{d_{j+1}^k}{2j} &= \frac{d_{j+1}^k}{2j+1} = d_j^k / 2 \\
a_{1,j}^k &= \frac{(d_j^k)^2 + (3d_{j-1}^k + d_{j+1}^k)d_j^k + 4d_{j-1}^k d_{j+1}^k}{8d_{j-1}^k (d_{j-1}^k + d_j^k + 1)} \\
a_{2,j}^k &= \frac{(d_j^k)^2 + (d_{j-1}^k + 3d_{j+1}^k)d_j^k + 4d_{j-1}^k d_{j+1}^k}{8d_{j+1}^k (d_{j-1}^k + d_j^k)} \\
a_{3,j}^k &= -\frac{(d_j^k)^2}{8d_{j+1}^k (d_j^k + d_j^k + 1)}
\end{align*}
\]
The NULI 4-pt scheme

> \( d^0_j \) starting parameters

At each step \( k \geq 0 \)

**REFINEMENT EQUATIONS**

\[
\begin{align*}
    p^{k+1}_{2j} &= p^k_j \\
    p^{k+1}_{2j+1} &= a^k_{0,j} p^k_{j-1} + a^k_{1,j} p^k_j + a^k_{2,j} p^k_{j+1} + a^k_{3,j} p^k_{j+2}
\end{align*}
\]

> \( d^{k+1}_{2j} = d^{k+1}_{2j+1} = \frac{d^k_j}{2} \) parameters updating

\[
\begin{align*}
    a^k_{0,j} &= -\frac{(d^k_j)^2}{8d^k_{j-1}(d^k_j-1+d^k_j)} \\
    a^k_{1,j} &= \frac{(d^k_j)^2 + (3d^k_{j-1}+d^k_{j+1})d^k_j + 4d^k_{j-1}d^k_{j+1}}{8d^k_{j-1}(d^k_j+d^k_{j+1})} \\
    a^k_{2,j} &= \frac{(d^k_j)^2 + (d^k_{j-1}+3d^k_{j+1})d^k_j + 4d^k_{j-1}d^k_{j+1}}{8d^k_{j+1}(d^k_j-1+d^k_j)} \\
    a^k_{3,j} &= -\frac{(d^k_j)^2}{8d^k_{j+1}(d^k_j+d^k_{j+1})}
\end{align*}
\]
Important remarks

- The centripetal parameterization \( \{x_j^k\}_{j \in \mathbb{Z}} \) for the point set \( \{p_j^k\}_{j \in \mathbb{Z}} \) is not recomputed at each step.
Important remarks

- The centripetal parameterization \( \{x^k_j\}_{j \in \mathbb{Z}} \) for the point set \( \{p^k_j\}_{j \in \mathbb{Z}} \) is not recomputed at each step.

Starting parameters \( d^0_j \) are simply updated through the formula

\[
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Important remarks

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→ the NULI 4-pt scheme is LINEAR!
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- the NULI 4-pt scheme is LINEAR!

- When all \( d_j^0 \) are *equal*, the NULI 4-pt scheme becomes the *uniform* 4-pt:

\[
a_{0,j}^k = a_{3,j}^k = -\frac{1}{16}, \quad a_{1,j}^k = a_{2,j}^k = \frac{9}{16}
\]
Properties of the NULI 4-pt scheme

QUADRATIC NULICS

NULI 4-POINT

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- Local support \([x_{j-2}, x_{j+2}]\)
  
  \([-2,2]\) in the uniform case

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- Local support \([x_{j-3}, x_{j+3}]\)
  
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- polynomials reproduction up to deg-2, also starting from non-equispaced samples

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- \(C^1\) smoothness
- polynomials reproduction:
  - up to deg-2 starting from non-equispaced samples
  - up to deg-3 starting from equispaced samples
Proposition 1

The basis function for the NULI 4-pt scheme has local support \([x_{j-3}, x_{j+3}]\).

Proof.

At step \(k = 0\) the support width is \(\sigma = [x_{j-2}, x_{j+2}]\).

At each successive step it is extended by

\[
\frac{x_{j-2} - x_{j-3}}{2^k} \quad \text{and} \quad \frac{x_{j+3} - x_{j+2}}{2^k}
\]

on the left and right side respectively.

Thus, after \(N\) steps it will be

\[
\sigma = \left[ x_{j-2} - \sum_{k=1}^{N} \frac{x_{j-2} - x_{j-3}}{2^k}, x_{j+2} + \sum_{k=1}^{N} \frac{x_{j+3} - x_{j+2}}{2^k} \right]
\]

and therefore when \(N \to +\infty\) \(\sigma = [x_{j-3}, x_{j+3}]\).
Support

**UNIFORM KNOTS**

\[ \sigma = [-3, 3] \]

**NON-UNIFORM KNOTS**

\[ \sigma = [x_{j-3}, x_{j+3}] \]
Proposition 2

The NULI 4-pt scheme generates $C^1$ limit curves for any choice of initial knots.

Proof.

After a few rounds of subdivision, we come to the knot intervals configuration

$$\ldots, a, a, b, b, b, \ldots \quad (a, b > 0)$$

which corresponds to the eigenanalysis of the local subdivision matrix

$$M = \begin{bmatrix} \frac{-1}{16} & \frac{9}{16} & \frac{9}{16} & \frac{-1}{16} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{4a+5b}{8(a+b)} & \frac{2a+7b}{16b} & -\frac{a^2}{8b(a+b)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b^2}{8a(a+b)} & \frac{7a+2b}{16a} & \frac{5a+4b}{8(a+b)} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{bmatrix}.$$
Smoothness analysis

- **Eigenvalues** of $M$: $\lambda_0 = 1$, $\lambda_1 = \frac{1}{2}$, $|\lambda_i| < \frac{1}{2}$ $\forall i \geq 2$

- **Right eigenvectors** for $\lambda_0$ and $\lambda_1$: $v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1/(ab) \\ 2/(ab) \\ 3/(ab) \end{bmatrix}$

The characteristic map $\psi[s, x]$ (where $s = 0, 1$ enumerates the two sectors identified around the EV) is the scalar limit function associated with $v_1$. Because $\psi[0, x] = -x$ and $\psi[1, x] = x/(ab)$ for $x > 0$, thus $\psi[0, x]$ and $\psi[1, x]$ cover respectively the negative and the positive portion of the parameter line in a 1-1 manner. Therefore $\psi$ is regular (i.e. it is a 1-1 and onto covering of the parameter line). This proves $C^1$ continuity of the associated scheme.
Proposition 3

The NULI 4-pt scheme can reproduce

- the set $\Pi_2$ of polynomials up to deg-2 starting from non-equispaced samples
- the set $\Pi_3$ of polynomials up to deg-3 starting from equispaced samples.

Proof.

The result follows from the fact that, starting with a point set $P^0 \in \Pi_2$, at each level $k \geq 0$ we compute $P^{k+1}$ by evaluating the NULICS interpolant with basis $\phi_{j,2}$ on knots $x_j^k$, at $\frac{x_j^k + x_{j+1}^k}{2}$.

$\Pi_3$ can be reproduced only when starting from equispaced samples because in this case the refinement rules become those of the classical 4-pt scheme.
Example 1

NULI 4-pt limit curve

NULICS quadratic interpolant
Example 1

NULI 4-pt limit curve

NULICS quadratic interpolant
Example 1: interpolation curves comparison

NULI 4-point limit curve
NULICS quadratic interpolant

🔗 the subdivision curve approximates the initial polyline more closely!
Example 2

NULI 4-pt limit curve

NULICS quadratic interpolant
Example 2

NULI 4-pt limit curve

NULICS quadratic interpolant
Example 2: interpolation curves comparison

NULI 4-point limit curve
NULICS quadratic interpolant

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- **a new idea** to construct spline-based high-quality $2n$-point interpolatory subdivision schemes;
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The new family of non-uniform interpolatory schemes

- includes $2n$-point refinement rules designed from deg-$n$ spline interpolants;
The NULI 6-point scheme

- cubics reproduction from non-uniform samples
- support $\sigma = [x_{j-5}, x_{j+5}]$
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The new family of non-uniform interpolatory schemes

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- establishes a fundamental step towards the construction of a spline-quality interpolatory scheme for surfaces of arbitrary topology

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Non-uniform local interpolatory subdivision surfaces (NULISS)

initial mesh  

NULISS  

section polyline  

section curve
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Bibliography


Thank you!