Non-uniform interpolatory subdivision designed from splines

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Outline

1. Motivations
2. Non-uniform local interpolatory cardinal splines (NULICS)
3. The NULI 4-pt scheme
   - Construction
   - Refinement rules
   - Main properties
     - Support
     - Smoothness
     - Polynomial reproduction
4. Application examples
5. Conclusions

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THE MOST EFFECTIVE METHODS FOR CURVE AND SURFACE DESIGN
generate a shape that

- passes through all the vertices of a given control net
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Proposed HERE:

interpolatory subdivision methods
designed from a class of
non-uniform local interpolatory
cardinal splines (NULICS)

Non-uniform local interpolatory cardinal splines (NULICS)

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  locally-supported fundamental splines leading to highly accurate local interpolation methods
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- C.K. Chui and J.M. De Villiers (1996):
  construction of general-degree NULICS and explicit formulation of their coefficients in terms of the data points
  - ➥ no solution of linear systems is required

Non-uniform local interpolatory cardinal splines (NULICS)

Main features of the degree-n local interpolatory cardinal spline basis:

- arbitrary knots $\{x_j\}_{j\in\mathbb{Z}}$
- compact support $[x_{j-n}, x_{j+n}]$
- C^{n-1} continuity
- polynomials reproduction up to degree n, starting from arbitrarily non-equispaced samples

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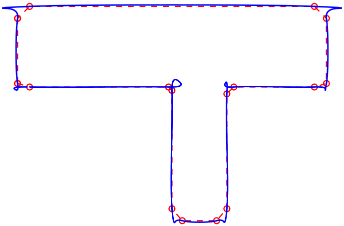
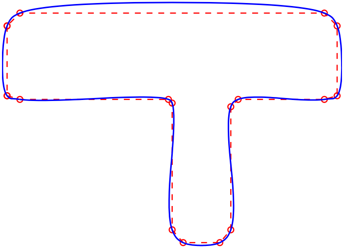
☛ No application example of the very general non-uniform case (probably due to its involved and complex representation)

NULICS: effectiveness of the local interpolatory method

Quadratic cardinal spline interpolation



UNIFORM KNOTS



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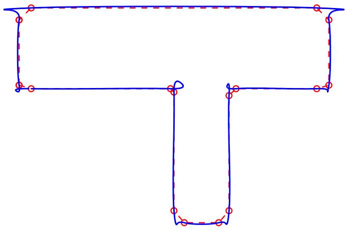

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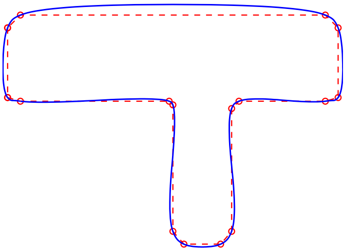

NON-UNIFORM KNOTS

CENTRIPETAL PARAMETERIZATION

- M. Floater (2008): advantages of the centripetal parameterization in spline interpolation
- N. Dyn, M. Floater, K. Hormann (2007): introduction of non-uniform parameters in interpolatory 4-point subdivision

NULICS: subdivision schemes design

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- **In this talk: quadratic NULICS $\Rrightarrow$ NULI 4-point subdivision**

The deg-2 NULICS basis

Step 1: *define the knot-partition*

x_{j-2} x_{j-1} x_j x_{j+1} x_{j+2}

x_{j+h} $(h = -2, ..., 2)$ arbitrary **break points** (parameters for points to be interpolated)

The deg-2 NULICS basis

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x_{j-2}		x_{j-1}		x_j		x_{j+1}		x_{j+2}
t_{j-4}	t_{j-3}	t_{j-2}	t_{j-1}	t_j	t_{j+1}	t_{j+2}	t_{j+3}	t_{j+4}

$t_{j+2h} \equiv x_{j+h} \quad (h = -2, \ldots, 2)$ **knots for interpolation points**

$t_{j+2h+1} = \frac{x_{j+h}+x_{j+h+1}}{2} \quad (h = -2, \ldots, 1)$ **intermediate knots**

} **complete knot sequence**

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$$\begin{array}{ccccccccc} x_{j-2} & & x_{j-1} & & x_j & & x_{j+1} & & x_{j+2} \\ t_{j-4} \underset{d_{j-2}}{\sqcup} & t_{j-3} \underset{d_{j-2}}{\sqcup} & t_{j-2} \underset{d_{j-1}}{\sqcup} & t_{j-1} \underset{d_{j-1}}{\sqcup} & t_j \underset{d_j}{\sqcup} & t_{j+1} \underset{d_j}{\sqcup} & t_{j+2} \underset{d_{j+1}}{\sqcup} & t_{j+3} \underset{d_{j+1}}{\sqcup} & t_{j+4} \end{array}$$

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Step 2: *represent the deg-2 NULICS basis centered at t_j in terms of B-spline basis functions*

$$\phi_{j,2}(x) = \sum_{i=0}^{5} b_i N_{i+j-4,2}(x)$$

where

$$b_0 = -\frac{(d_{j-2})^2}{4d_{j-1}(d_{j-2}+d_{j-1})} \qquad b_1 = \frac{d_{j-2}}{4(d_{j-2}+d_{j-1})} \qquad b_2 = \frac{d_{j-1}+3d_j}{4d_j}$$

$$b_3 = \frac{3d_{j-1}+d_j}{4d_{j-1}} \qquad b_4 = \frac{d_{j+1}}{4(d_j+d_{j+1})} \qquad b_5 = -\frac{(d_{j+1})^2}{4d_j(d_j+d_{j+1})}$$

and $N_{i+j-4,2}(x)$ is the quadratic B-spline with support $[t_{i+j-4}, t_{i+j-1}]$.

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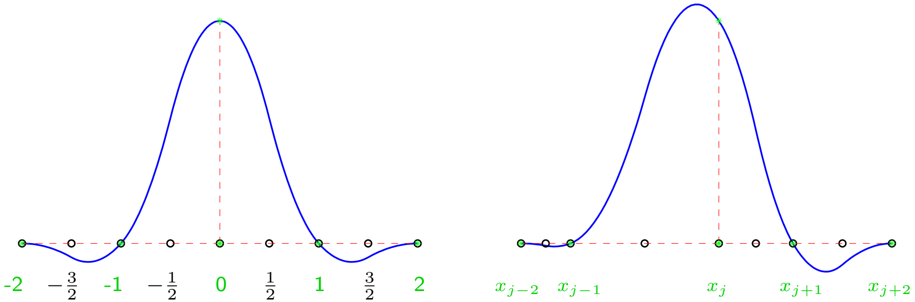
$$\phi_{j,2}(x_j) = 1, \qquad \phi_{j,2}(x_{j+h}) = 0 \quad \forall h \neq 0$$



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Towards a NULI 4-pt scheme

- Let p^0_{j+h} (h=-1,0,1,2) be the quadruple of starting points

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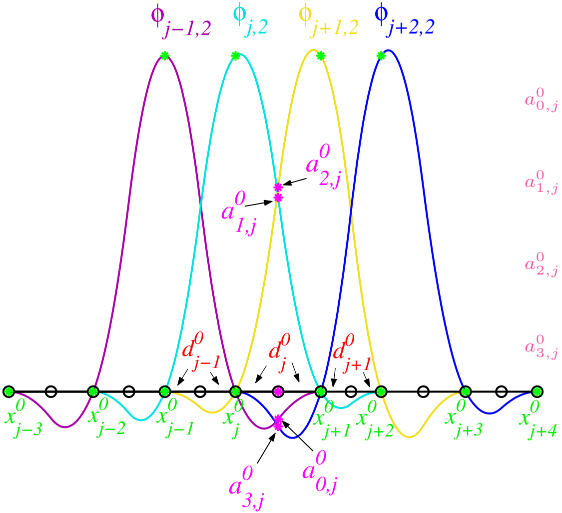

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The NULI 4-pt scheme

- d_j^0 starting parameters

At each step $k \geq 0$

REFINEMENT EQUATIONS

$$p_{2j}^{k+1} = p_j^k$$
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Coefficients $a_{0,j}^k, \ldots, a_{3,j}^k$ coincide with the values of the deg-2 basis functions $\phi_{j-1,2}, \ldots, \phi_{j+2,2}$ defined on the refined knot-partition with $d_j^0/2^k$-length intervals, at the central knot $\frac{x_j^k + x_{j+1}^k}{2}$.

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➢ $$\boxed{d_{2j}^{k+1} = d_{2j+1}^{k+1} = \frac{d_j^k}{2}}$$ parameters updating

Important remarks

- The centripetal parameterization $\{x_j^k\}_{j\in\mathbb{Z}}$ for the point set $\{p_j^k\}_{j\in\mathbb{Z}}$ is *not* recomputed at each step.

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  ➠ **the NULI 4-pt scheme is LINEAR!**

- When all d_j^0 are *equal*, the NULI 4-pt scheme becomes the *uniform* 4-pt:

$$a_{0,j}^k = a_{3,j}^k = -\frac{1}{16}, \quad a_{1,j}^k = a_{2,j}^k = \frac{9}{16}$$

Properties of the NULI 4-pt scheme

QUADRATIC NULICS

NULI 4-POINT

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QUADRATIC NULICS

- local support $[x_{j-2}, x_{j+2}]$ ([-2,2] in the uniform case)

NULI 4-POINT

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- polynomials reproduction up to deg-2, also starting from *non-equispaced* samples

NULI 4-POINT

- ➥ local support $[x_{j-3}, x_{j+3}]$ ([-3,3] in the uniform case)
- ➥ C^1 smoothness
- ➥ polynomials reproduction:
  - up to deg-2 starting from *non-equispaced* samples
  - up to deg-3 starting from *equispaced* samples

Support

Proposition 1

The basis function for the NULI 4-pt scheme has local support $[x_{j-3}, x_{j+3}]$.

Proof.

At step $k = 0$ the support width is $\sigma = [x_{j-2}, x_{j+2}]$.
At each successive step it is extended by

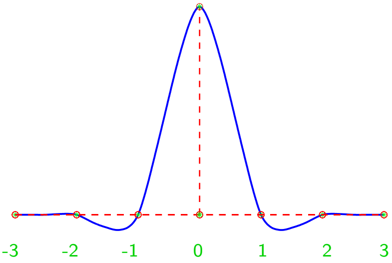
$$\frac{x_{j-2} - x_{j-3}}{2^k} \quad \text{and} \quad \frac{x_{j+3} - x_{j+2}}{2^k}$$

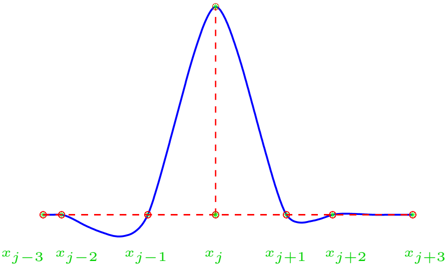
on the left and right side respectively.
Thus, after N steps it will be

$$\sigma = \left[x_{j-2} - \sum_{k=1}^{N} \frac{x_{j-2} - x_{j-3}}{2^k}, x_{j+2} + \sum_{k=1}^{N} \frac{x_{j+3} - x_{j+2}}{2^k} \right]$$

and therefore when $N \to +\infty$ $\sigma = [x_{j-3}, x_{j+3}]$. □

Support





UNIFORM KNOTS

$\Downarrow$

$$\sigma = [-3, 3]$$

NON-UNIFORM KNOTS

$\Downarrow$

$$\sigma = [x_{j-3}, x_{j+3}]$$

Smoothness analysis

Proposition 2

The NULI 4-pt scheme generates C^1 limit curves for any choice of initial knots.

Proof.

After a few rounds of subdivision, we come to the knot intervals configuration

$$\ldots,\ a,\ a,\ b,\ b,\ b,\ \ldots \qquad (a, b > 0)$$

which corresponds to the eigenanalysis of the local subdivision matrix

$$M = \begin{bmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{4a+5b}{8(a+b)} & \frac{2a+7b}{16b} & -\frac{a^2}{8b(a+b)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b^2}{8a(a+b)} & \frac{7a+2b}{16a} & \frac{5a+4b}{8(a+b)} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{bmatrix}.$$

Smoothness analysis

- *Eigenvalues* of M: $\lambda_0 = 1, \ \lambda_1 = \frac{1}{2}, \ |\lambda_i| < \frac{1}{2} \ \forall i \geq 2$
- *Right eigenvectors* for λ_0 and λ_1: $v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v_1 = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1/(ab) \\ 2/(ab) \\ 3/(ab) \end{bmatrix}$

The characteristic map $\psi[s, x]$ (where $s = 0, 1$ enumerates the two sectors identified around the EV) is the scalar limit function associated with v_1. Because $\psi[0, x] = -x$ and $\psi[1, x] = x/(ab)$ for $x > 0$, thus $\psi[0, x]$ and $\psi[1, x]$ cover respectively the negative and the positive portion of the parameter line in a 1-1 manner. Therefore ψ is *regular* (i.e. it is a 1-1 and onto covering of the parameter line). This proves C^1 continuity of the associated scheme. □

Polynomial reproducibility

Proposition 3

The NULI 4-pt scheme can reproduce

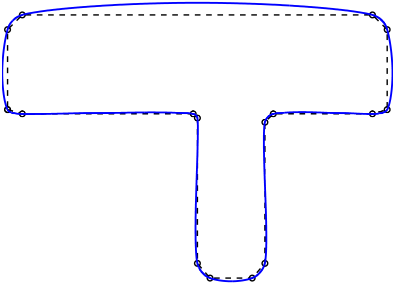
- the set Π_2 of polynomials up to deg-2 starting from non-equispaced samples
- the set Π_3 of polynomials up to deg-3 starting from equispaced samples.

Proof.

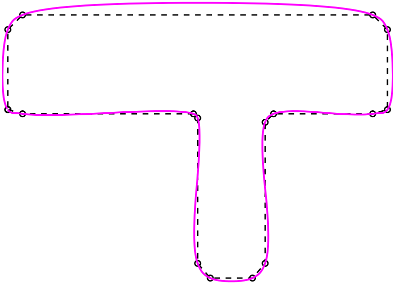
The result follows from the fact that, starting with a point set $P^0 \in \Pi_2$, at each level $k \geq 0$ we compute P^{k+1} by evaluating the NULICS interpolant with basis $\phi_{j,2}$ on knots x_j^k, at $\frac{x_j^k + x_{j+1}^k}{2}$.

Π_3 can be reproduced only when starting from equispaced samples because in this case the refinement rules become those of the classical 4-pt scheme. □

Example 1

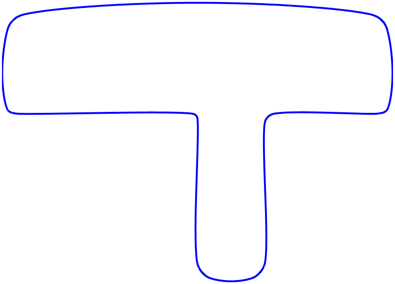


NULI 4-pt limit curve



NULICS quadratic interpolant

Example 1



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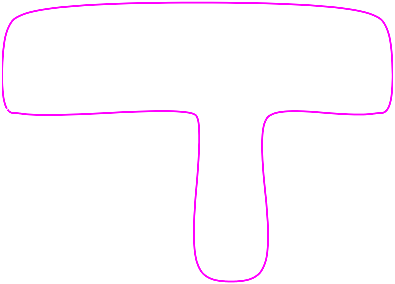
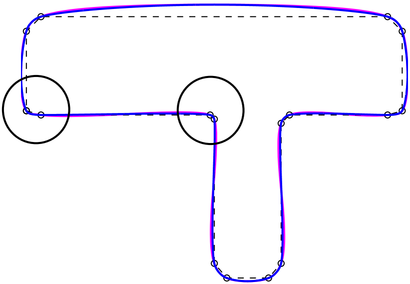
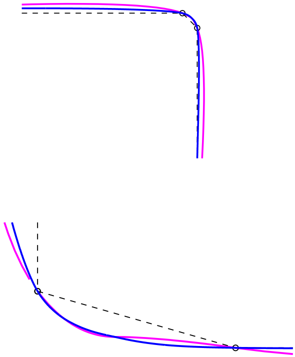


NULICS quadratic interpolant

Example 1: interpolation curves comparison





NULI 4-point limit curve
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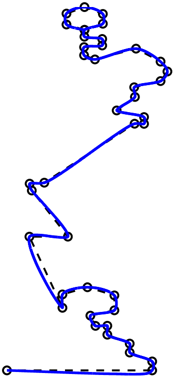
➫ the subdivision curve approximates the initial polyline more closely!

Example 2



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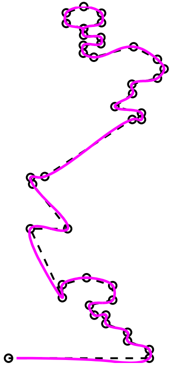

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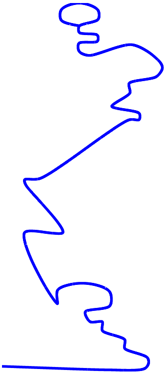

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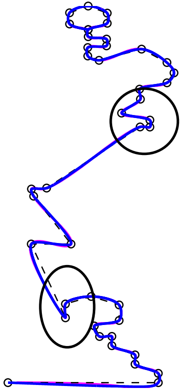

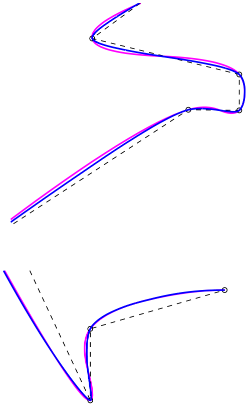

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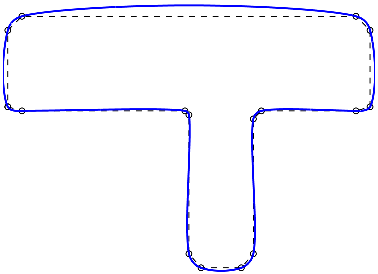
The NULI 6-point scheme



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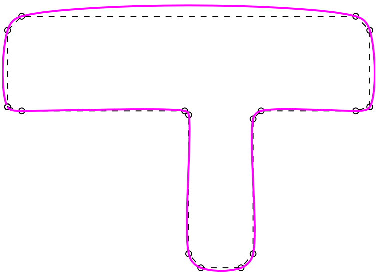

NULICS cubic interpolant

- cubics reproduction from non-uniform samples
- support $\sigma = [x_{j-5}, x_{j+5}]$
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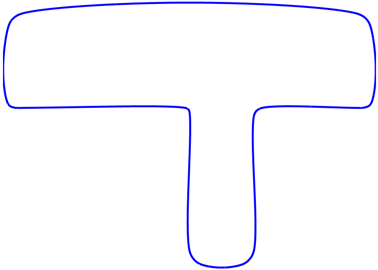

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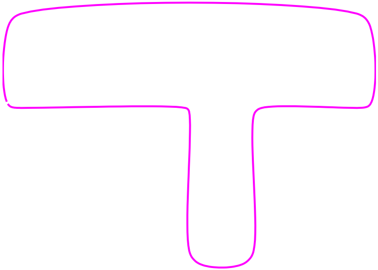

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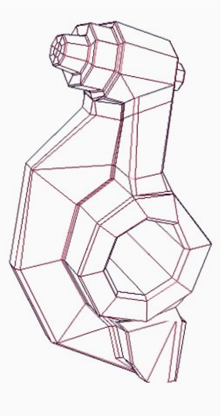
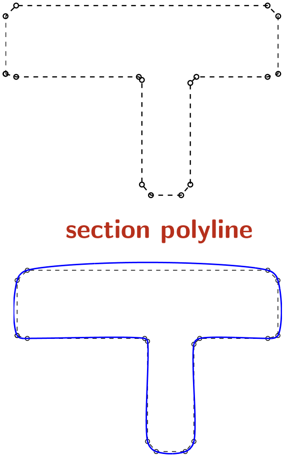
- ☛ establishes a fundamental step towards the construction of a spline-quality interpolatory scheme for surfaces of arbitrary topology
  ↝ SIMAI Conference, Rome (Italy) - September '08

Non-uniform local interpolatory subdivision surfaces (NULISS)



initial mesh

NULISS



section polyline

section curve

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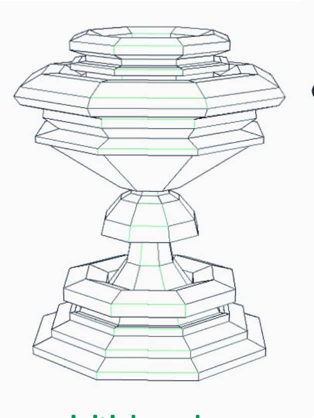

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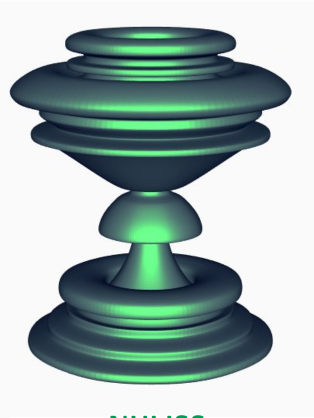

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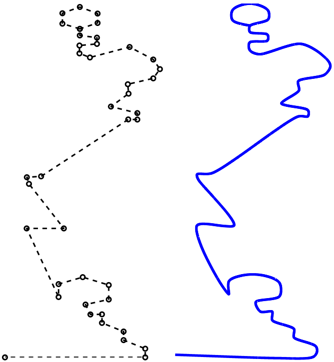

section

polyline curve

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Thank you!